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One-complemented subspaces of Musielak–Orlicz sequence spaces

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Abstract

The aim of this paper is to characterize one-complemented subspaces of finite codimension in the Musielak–Orlicz sequence space l_Φ. We generalize the well-known fact (Ann. Mat. Pura Appl. 152 (1988) 53; Period. Math. Hungar. 22 (1991) 161; Classical Banach Spaces I, Springer, Berlin, 1977) that a subspace of finite codimension in l_p, $1 \leqslant p < \infty$, is one-complemented if and only if it can be expressed as a finite intersection of kernels of functionals with at most two coordinates different from zero. Under some smoothness condition on $\Phi = (\phi_n)$ we prove a similar characterization in l_Φ. In the case of Orlicz spaces we obtain a complete characterization of one-complemented subspaces of finite codimension, which extends and completes the results in Randrianantoanina (Results Math. 33(1–2) (1998) 139). Further, we show that the well-known fact that a one-complemented subspace of finite codimension in l_p, $1 \leqslant p < \infty$, is an intersection of one-complemented hyperplanes, is no longer valid in Orlicz or Musielak–Orlicz spaces. In the last section we characterize l_p-spaces, $1 < p < \infty$, and separately l_2-spaces, in terms of one-complemented hyperplanes, in the class of Musielak–Orlicz and Orlicz spaces as well.

Keywords: One-complemented subspaces; Musielak–Orlicz sequence spaces; Minimal projections

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0. Introduction

We adopt here the standard notations from Banach space theory. Given Banach spaces X and Y by $\mathcal{L}(X, Y)$ we denote the space of all linear bounded operators from X to Y, and if $X = Y$ then it is denoted by $\mathcal{L}(X)$. As usual symbol $\ker(T)$ is reserved for the kernel of a linear operator T, T^* for a conjugate operator of T, and Id for an identity operator. If Y is a closed subspace of a Banach space X, $P \in \mathcal{L}(X, Y)$ is called a *projection* whenever $P|_Y = Id$, that is $P^2 = P$. The set of all projections from X onto Y will be further denoted by $\mathcal{P}(X, Y)$. It is clear that if $Y \neq \{0\}$, then for any $P \in \mathcal{P}(X, Y)$, $\|P\| \geqslant 1$. A subspace Y of a Banach space X is called *one-complemented* if there exists $P \in \mathcal{P}(X, Y)$ such that $\|P\| = 1$.

One-complemented subspaces Y of a Banach space X have the nice property that any operator from $\mathcal{L}(Y)$ has a linear extension onto the whole of X with the same norm. It is easy to see and very well known that any closed subspace Y of a Hilbert space H is one-complemented and that in any Banach space, each one-dimensional subspace is one-complemented. In general however, there are not too "many" one-complemented subspaces. For example, by the classical Kakutani theorem (see e.g. [1]), a Banach space X of dimension $\geqslant 3$ which is not a Hilbert space has a two-dimensional subspace and a hyperplane which are not one-complemented. Also in [3] it has been shown that if $1 < p < \infty$, $p \neq 2$, and (Ω, Σ, μ) is a nonatomic σ-finite measure space, then there is no one-complemented subspace of $L_p(\Omega, \Sigma, \mu)$ of finite codimension. The same result is true for real separable rearrangement-invariant spaces on [0, 1] not isometric to L_2 [20].

In the case of sequence spaces, the situation is different. By a result of Bohnenblust [5], a linear subspace Y of $l_p^{(n)}$ ($\mathbb{R}^n$ with the l_p-norm), $1 < p < \infty$, $p \neq 2$, is one-complemented if and only if Y can be represented as an intersection of kernels of functionals having at most two coordinates different than zero. It has been later generalized to infinite dimensional spaces l_p, $1 \leqslant p < \infty$, and c_0. In fact, following Theorem 2.a.4. in [16], any one-complemented subspace of l_p or c_0 is the closure of a linear span of disjointly supported elements, which easily implies the necessary part of the Bohnenblust result, that every one-complemented subspace of l_p or c_0 is an intersection of kernels of functionals in $l_{p'}$, $1/p' = 1 - 1/p$, or l_1, respectively, with at most two coordinates different than zero (see also [6,7]). One-complemented hyperplanes in l_1 and c_0 have been completely described in [4], and projections onto subspaces of finite codimension in l_∞ have been considered in [2]. For results concerning more general sequence spaces like Orlicz or Lorentz sequence spaces see [19,20] and references there.

The aim of this paper is to study one-complemented subspaces of finite codimension in Musielak–Orlicz sequence spaces. We consider here only the real case.

Preliminaries contain basic facts on projections and Musielak–Orlicz sequence spaces, as well as some technical definitions and results that will be of use later.

The main results of the paper are contained in Section 2. In Theorem 2.7 we present, under a smoothness condition (S) introduced in the preliminaries, a complete characterization of one-complemented subspaces of Musielak–Orlicz sequence spaces of finite codimension in terms of so called *proper representation* of these subspaces (see Definition 1.7). This characterization in the case of Orlicz spaces has a simpler form and is stated in Theorem 2.10 and Corollaries 2.11 and 2.12. These results are extensions to Musielak–Orlicz spaces including nonseparable ones, of the known characterization in Orlicz separable spaces given in [19]. They are proved using a different technique than in [19], which allows us to remove the assumption made there, that a basic vector belongs to the subspace of finite codimension. It should be pointed out however that our technique requires a certain smooth assumption on Φ (condition (S)), which limits the class of Musielak–Orlicz functions generating the spaces. On the other hand condition (S) is not so restrictive: Proposition 1.4 shows that for an arbitrary Musielak–Orlicz space l_Φ there exists a Musielak–Orlicz space l_Ψ isomorphic to l_Φ and such that Ψ satisfies condition (S). We finish the section with some corollaries on one-complemented hyperplanes in l_Φ and with a new proof of a characterization of one-complemented subspaces of finite codimension in l_p, $1 < p < \infty$, given in [6].

In Section 3 we show that a finite intersection of one-complemented hyperplanes in l_Φ must also be a one-complemented subspace (Theorem 3.1). We also provide (Theorems 3.2 and 3.3) examples of both Orlicz and Musielak–Orlicz spaces such that the converse statement does not hold. It shows that the result true in l_p, $1 \leqslant p < \infty$, that a subspace of finite codimension is one-complemented if and only if it is an intersection of one-complemented hyperplanes [6,7], cannot be extended to Orlicz and thus also to Musielak–Orlicz spaces.

In Section 4, a characterization of l_p-spaces, $1 < p < \infty$, (Theorem 4.1) and l_2-spaces (Theorem 4.5) in the class of Musielak–Orlicz spaces, hence in Orlicz spaces as well, is given in terms of one-complemented hyperplanes.

1. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ stand for the natural numbers, integers and real numbers, respectively. Let $(X, \|\ \|)$ be a Banach space and let $0 \neq x \in X$. A functional $f \in X^*$ of norm one is called *a norming functional* (or *supporting functional*) of *x* whenever $f(x) = \|x\|$. Recall also that $0 \neq x \in X$ is said to be a *smooth* point whenever its supporting functional is unique. We say that *X* is *smooth* if every element of its unit sphere is smooth. If *Y* is a nonempty subset of *X* then

$$Y^\perp = \{f \in X^* : f|_Y = 0\}.$$

By $\mathrm{span}[Y]$ we denote the linear subspace of *X* spanned by *Y*.

The first two lemmas, crucial in our investigations, are well known. We include their proofs here for the sake of completeness.

Lemma 1.1. *Let Y be a closed subspace of a Banach space X. Then $P \in \mathcal{P}(X, Y)$ has norm one if and only if for each $0 \neq y \in Y$ there exists an element f of $(\ker(P))^\perp$, which is a norming functional for y. If $y \in Y$ is a smooth point of X then f is uniquely determined.*

Proof. We first observe that $\|P\| = 1$ if and only if for any $y \in Y$, zero is the best approximation to *y* in $V = \ker(P)$. Indeed, if $\|P\| = 1$ then for any $x \in X$,

$$\|x - (Id - P)x\| \leqslant \|P\| \mathrm{dist}(x, V) \leqslant \|x - (Id - P)x\|,$$

which implies that $\operatorname{dist}(y, V) = \|y\|$ for any $y \in Y$. Conversely, if zero is the best approximation to $y = Px$ in V, $x \in X$, then $(Id - P)x$ is the best approximation to x in V. Hence for any $x \in X$,

$$\|Px\| = \|x - (Id - P)x\| \leqslant \|x\|$$

and so $\|P\| = 1$.

Recall also that given a subspace V of X and $x \in X \setminus V$ with $\operatorname{dist}(x, V) = d > 0$, $v \in V$ is the best approximation to x in V if and only if there exists $f \in V^{\perp}$ which is a norming functional for $x - v$.

We complete the proof by applying the above observations to $V = \ker(P)$, $v = 0$ and $0 \neq y \in Y$. □

The next result provides a representation of a projection on a subspace of finite codimension (see e.g. [4]). Let, as usual, $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Lemma 1.2. *Assume X is a normed space and let $Y \subset X$ be a subspace of codimension n. Let $\{f_1, \ldots, f_n\} \in Y^{\perp}$ be a basis of $Y^{\perp}$, and suppose $P \in \mathcal{P}(X, Y)$. Then there exist uniquely determined $z_1, \ldots, z_n \in \ker(P)$ such that*

$$f_i(z_j) = \delta_{ij}$$

and

$$Px = x - \sum_{i=1}^{n} f_i(x) z_i$$

for $x \in X$.

Proof. Since $Y^{\perp} = \operatorname{span}[f_1, \ldots f_n]$ and $X = Y \oplus \ker(P)$, for any $z \in \ker(P)$, if $f_i(z) = 0$ for $i = 1, \ldots, n$, then $z = 0$. This shows that $f_1|_{\ker(P)}, \ldots, f_n|_{\ker(P)}$ are linearly independent. Hence there exist uniquely determined $z_1, \ldots, z_n \in \ker(P)$ such that $f_i(z_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. Set for $x \in X$,

$$Qx = x - \sum_{i=1}^{n} f_i(x) z_i.$$

Note that $Q|_Y = P|_Y = Id|_Y$ and $Q|_{\ker(P)} = 0$, since $Qz_i = 0$ for $i = 1, \ldots, n$. Hence $Q = P$, which shows our claim. □

Also the following simple fact will be frequently used.

Lemma 1.3. *Let X, Z be two normed spaces and let $T : X \to Z$ be a linear surjective isometry. Then a subspace $Y \subset X$ is one-complemented in X if and only if $T(Y)$ is one-complemented in Z.*

Now we present some introductory facts on Musielak–Orlicz spaces. A function $\phi : \mathbb{R}_+ \to [0, +\infty)$ is said to be an *Orlicz function* if $\phi(0) = 0$, ϕ is strictly increasing and convex.

By ϕ^* we denote its conjugate function in the sense of Young, that is

$$\phi^*(u) = \sup_{v>0} \{uv - \phi(v)\}, \quad u \geqslant 0$$

and we notice that ϕ^* is an extended real-valued convex function. If $\phi(u) = (1/p)u^p$, $1 < p < \infty$, then $\phi^*(u) = (1/p')u^{p'}$, where $1/p + 1/p' = 1$. Further, a sequence $\Phi = (\phi_n)$ of Orlicz functions ϕ_n will be called a *Musielak–Orlicz function* whenever $\phi_n(1) = 1$ for every $n \in \mathbb{N}$. By $\Phi^* = (\phi_n^*)$ we will denote its conjugate function.

Let l_0 denote the space of all real-valued sequences. With each Musielak–Orlicz function Φ we can associate a mapping $\rho_\Phi : l_0 \to [0, +\infty]$ defined by

$$\rho_\Phi(x) = \sum_{n=1}^{\infty} \phi_n(|x_n|),$$

where $x = (x_n) \in l_0$. Given a Musielak–Orlicz function Φ, let l_Φ denote the corresponding *Musielak–Orlicz space*, that is

$$l_\Phi = \{x \in l_0 : \lim_{\lambda \to 0} \rho_\Phi(\lambda x) = 0\}.$$

If a sequence $\Phi = (\phi_n)$ is constant, that is $\phi_n = \phi$ for every $n \in \mathbb{N}$, then l_Φ is an *Orlicz sequence space* and further it will be denoted by l_ϕ. The space l_Φ equipped with the Luxemburg norm

$$\|x\| = \|x\|_\Phi := \inf\{\lambda > 0 : \rho_\Phi(x/\lambda) \leqslant 1\}$$

is a Banach space. Recall also that given Musielak–Orlicz functions $\Phi = (\phi_n)$ and $\Psi = (\psi_n)$, the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms if and only if Φ is equivalent to Ψ, that is, for some $K, \delta > 0$ and $(c_n) \in \ell_1^+$,

$$\phi_n(Ku) \leqslant \psi_n(u) + c_n, \quad \text{whenever} \quad \psi_n(u) \leqslant \delta$$

and

$$\psi_n(Ku) \leq \phi_n(u) + c_n, \quad \text{whenever} \quad \phi_n(u) \leqslant \delta.$$

Observe that the assumption $\phi_n(1) = 1$ for every $n \in \mathbb{N}$ is not a real restriction on Musielak–Orlicz function Φ. In fact, for every sequence $\Phi = (\phi_n)$, where ϕ_n are Orlicz functions, there exists a function $\Psi = (\psi_n)$ with $\psi_n(1) = 1$ and such that l_Φ is isometric to l_Ψ. It is enough to take $\psi_n(t) = \phi_n(a_n t)$, where $\phi_n(a_n) = 1$ for every $n \in \mathbb{N}$.

We will also consider here the finite dimensional spaces $l_\Phi^{(m)}$, defined on $\mathbb{R}^m$ analogously as l_Φ. The space $l_\Phi^{(m)}$ can be identified with the subspace of l_Φ consisting of all $x = (x_n) \in l_\Phi$ such that $x_n = 0$ for all $n \geqslant m + 1$.

If (f_i) is a sequence of elements f_i in l_Φ, then by f_{ij} we denote the coefficients of f_i, that is $f_i = (f_{ij})$.

An important subspace of l_Φ, called the *subspace of finite elements* and denoted by h_Φ is defined as

$$h_\Phi = \{x \in l_\Phi : \rho_\Phi(\lambda x) < \infty \text{ for any } \lambda > 0\}.$$

It is well known that h_Φ is a closed separable subspace of l_Φ with the one-unconditional basis consisting of the standard unit vectors $e_i = (0, \ldots, 1_i, 0, \ldots)$. It is easy to see that for every $x \in h_\Phi$, $\|x\| = 1$ if and only if $\rho_\Phi(x) = 1$. Moreover, $h_\Phi = l_\Phi$ if and only if either the dimension of l_Φ is finite or Φ satisfies a growth condition called δ_2 [11,12,16].

Recall that for every $y \in l_{\Phi^*}$, the functional

$$f_y(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_n) \in l_\Phi,$$

is bounded on $(l_\Phi, \| \ \|_\Phi)$ and is called a *regular functional*. We denote by R_Φ the set of all regular functionals on l_Φ. The spaces R_Φ and l_{Φ^*} are order isomorphic [10,21] and so by usual identification we often write $f_y = y$. We say that f_S is a *singular functional* on l_Φ whenever $f_S(x) = 0$ for every $x \in h_\Phi$. The set of all singular functionals on l_Φ will be denoted by S_Φ. It is well known [10, Lemma 1.1 and Theorem 2.9] that

$$(l_\Phi)^* = R_\Phi \oplus_1 S_\Phi.$$

More precisely, for each $f \in (l_\Phi)^*$ there exist uniquely determined $r(f) \in R_\Phi$ and $s(f) \in S_\Phi$ such that

$$f = r(f) + s(f)$$

and

$$\|f\| = \|r(f)\| + \|s(f)\|.$$

It is clear that the operators r and s are linear projections of $(l_\Phi)^*$ onto R_Φ and S_Φ, respectively. Since R_Φ and l_{Φ^*} are order isomorphic we will identify $r(f)$ with an element of l_{Φ^*}, and then $r(f) = (r(f)_n)$. More information on Musielak–Orlicz spaces one can find in [17,16, vol. I], [8,11,12, 21,22].

An Orlicz function ϕ is said to satisfy condition (s) whenever ϕ is differentiable on $[0, \infty)$, $\phi(1) = 1$ and both ϕ and ϕ' vanish only at zero. We say that ϕ satisfies condition (S) if ϕ fulfills (s), ϕ'' is continuous on $[0, \infty)$ and it vanishes only at zero. We also agree that a Musielak–Orlicz function $\Phi = (\phi_n)$ satisfies condition (s) or (S) whenever all ϕ_n satisfy (s) or (S), respectively. Notice that the assumption (s) on ϕ implies that ϕ' is already continuous (see [13, Theorem 1, p. 156]).

Condition (S) on Φ is not very restrictive. In fact we have the following result.

Proposition 1.4. *For any Musielak–Orlicz function $\Phi = (\phi_n)$ there exists a Musielak–Orlicz function $\Psi = (\psi_n)$ equivalent to Φ and satisfying condition (S). Consequently, the identity operator from l_Φ to l_Ψ is an isomorphism.*

Proof. Let $\chi_n(t) = \int_0^t \phi_n(u)/u\,du$ and $\gamma_n(t) = \int_0^t \chi_n(u)/u\,du$. It is clear that γ_n are Orlicz functions of class $C^2(0, \infty)$. Taking $d_n \in (0, 1)$ such that $\sum_{n=1}^{\infty} \gamma_n(8d_n) < \infty$, define

$$\psi_n(t) = \int_0^t \bar{\phi}_n(u)/u\,du,$$

where

$$\bar{\phi}_n(t) = \begin{cases} \gamma_n'(d_n)t^3/(3d_n^2) & \text{if } 0 \leqslant t \leqslant d_n, \\ \gamma_n'(d_n)d_n/3 + \gamma_n(t) - \gamma_n(d_n) & \text{if } t \geqslant d_n. \end{cases}$$

It is easy to see that both derivatives ψ_n' and ψ_n'' are continuous and they vanish only at zero. Thus $\Psi = (\psi_n)$ satisfies condition (S). Since ϕ_n are convex, both χ_n and γ_n are convex. Hence all quotients $\phi_n(t)/t,\ \chi_n(t)/t$ and $\gamma_n(t)/t$ are increasing functions with respect to $t > 0$. Therefore

$$\phi_n(t/2) \leqslant \int_{t/2}^{t} \phi_n(u)/u\, du \leqslant \chi_n(t) \leqslant \phi_n(t)$$

as well as

$$\chi_n(t/2) \leqslant \gamma_n(t) \leqslant \chi_n(t)$$

for all $t \geqslant 0$ and $n \in \mathbb{N}$. Hence

$$\gamma_n(t/2) \leqslant \phi_n(t/2) \leqslant \gamma_n(2t)$$

and so $\Gamma = (\gamma_n)$ and $\Phi = (\phi_n)$ are equivalent. Since $\bar{\phi}_n(u)/u$ is increasing, ψ_n are convex and by the similar argument as above we get

$$\bar{\phi}_n(t/2) \leqslant \psi_n(t) \leqslant \bar{\phi}_n(t)$$

for all $n \in \mathbb{N}$ and $t \geqslant 0$. We also have for $t \geqslant d_n$,

$$\bar{\phi}_n(t) \leqslant \gamma_n'(d_n)d_n + \gamma_n(t) \leqslant \gamma_n'(2d_n) + \gamma_n(t) \leqslant \gamma_n(2t) + \gamma_n(t) \leqslant 2\gamma_n(2t)$$

and for $t \geqslant 2d_n$,

$$\bar{\phi}_n(t) \geqslant \gamma_n(t) - \gamma_n(d_n) \geqslant \gamma_n(t) - \gamma_n(t/2) \geqslant (1/2)\gamma_n(t).$$

Therefore, for $n \in \mathbb{N},\ t \geqslant 2d_n$,

$$(1/2)\gamma_n(t) \leqslant \bar{\phi}_n(t) \leqslant 2\gamma_n(2t).$$

It follows that for $t \geqslant 2d_n,\ n \in \mathbb{N}$,

$$(1/2)\gamma_n(t) \leqslant \bar{\phi}_n(t) \leqslant \psi_n(2t) \leqslant \bar{\phi}_n(2t) \leqslant 2\gamma_n(4t).$$

Thus for any $n \in \mathbb{N}$ and $t \geqslant 0$,

$$\gamma_n(t) \leqslant 2\psi_n(2t) + \gamma_n(2d_n) \quad \text{and} \quad \psi_n(2t) \leqslant 2\gamma_n(4t) + \psi_n(4d_n) \leqslant 2\gamma_n(4t) + 2\gamma_n(8d_n).$$

But $\sum_{n=1}^{\infty} \gamma_n(8d_n) < \infty$, which shows that $\Gamma = (\gamma_n)$ and $\Psi = (\psi_n)$ are equivalent. Since Γ is equivalent to Φ, the proof is complete. □

The following description (see [10, Lemma 1.7 and Theorem 1.9]) of smooth points and supporting functionals in l_Φ will play an essential role in our investigations.

Theorem 1.5. *Let $\Phi = (\phi_n)$ be a Musielak–Orlicz function satisfying condition (s). Then each $0 \neq x = (x_n) \in h_\Phi$ is a smooth point in l_Φ. Moreover, the supporting functional of $0 \neq x = (x_n) \in h_\Phi$ is a regular functional f_y determined by $y = N(x)/C_x \in l_{\Phi^*}$, where*

$$N(x) = ((\operatorname{sgn} x_n)\phi_n'(|x_n|/\|x\|)$$

and $C_x = \sum_{n=1}^\infty |x_n|\phi_n'(|x_n|/\|x\|)$

From Lemma 1.1 and Theorem 1.5 one can deduce the following result.

Corollary 1.6. *Let Φ satisfy condition (s) and let $Y \subset l_\Phi$ be a closed subspace of l_Φ. If $P \in \mathcal{P}(l_\Phi, Y)$ is a projection of norm one, then for every $0 \neq y \in Y \cap h_\Phi$ and $z \in \ker(P)$, $N(y)(z) = 0$. Moreover, if $l_\Phi = h_\Phi$ then the converse implication also holds true.*

Henceforth in the paper all Orlicz and Musielak–Orlicz functions which are considered are required to satisfy at least condition (s).

Finally let us agree that for given sequence $x = (x_n)$, $\operatorname{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\}$ is its support and for $j \in \mathbb{N}$,

$$P_j x = (x_1, \ldots, x_j, 0, \ldots) \quad \text{and} \quad (Id - P_j)x = (0, \ldots, 0, x_{j+1}, x_{j+2}, \ldots).$$

The following definition that "normalizes" the representation $\{f_1, \ldots, f_n\}$ of a subspace of codimension *n* given in Lemma 1.2, allows us to formulate many further results in less technical ways.

Definition 1.7. Let $Y \subset l_\Phi$ or $Y \subset l_\Phi^{(m)}$ be a subspace of codimension n. Let $k = \dim(r(Y^\perp))$. Then $F = \{f_1, \ldots, f_n\} \subset Y^\perp$ is called *a proper representation of Y* if the following conditions are satisfied.

(1) F is linearly independent, $\operatorname{span}[F] = Y^\perp$ and $r(f_i) = 0$ for $i \geqslant k+1$ and $k < n$.

(2) $r(f_i)_j = \delta_{ij}$ for $i, j = 1, \ldots, k$.

In particular, if $l_\Phi = h_\Phi$ or Y is a subspace of a finite dimensional space $l_\Phi^{(m)}$, then all functionals are regular, and so $k = n$ and $r(f_i) = f_i$.

Lemma 1.8. *Let $Y \subset l_\Phi$ or $Y \subset l_\Phi^{(m)}$ be a subspace of codimension n. Then up to isometry of l_Φ or $l_\Phi^{(m)}$, there exists $F \subset Y^\perp$ which is a proper representation of Y.*

Proof. Set $k = \dim(r(Y^\perp))$. If $k = 0$ then any basis of $Y^\perp$ is a proper representation of Y. If $k > 0$, choose $F_1 = \{f_1, \ldots, f_k\} \subset Y^\perp$ such that $r(Y^\perp) = \operatorname{span}[r(F_1)]$. If $k < n$, let $F_2 = \{f_{k+1}, \ldots, f_n\}$ be any basis of $Y^\perp \cap \ker(r)$ ($F_2 = \emptyset$ if $k = n$). Put $F_3 = F_1 \cup F_2$. Observe that $Y^\perp = \operatorname{span}[F_3]$ and $r(f_i) = 0$ for $i = k+1, \ldots, n$ if $k < n$. Since $r(f_1), \ldots, r(f_k)$ are linearly independent, there exists $m_1 < \cdots < m_k$ such that $\det[r(f_j)_{m_i}]_{i,j=1,\ldots,k} \neq 0$. By a permutation of integers, to which there corresponds a permutation of the sequence (ϕ_n) and an isometric isomorphism of l_Φ, it may be supposed that $m_i = i$, $i = 1, \ldots, k$. In fact observe that any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ induces a linear isometry $T_\sigma : l_\Phi \to l_{\Phi_\sigma}$ defined as $T_\sigma x = (x_{\sigma(n)})$, where $\Phi_\sigma = (\phi_{\sigma(n)})$. Hence for any

$i = 1, \ldots, k$, there exists $a_{i1}, \ldots, a_{ik}$ such that

$$\sum_{l=1}^{k} a_{il} r(f_l)_j = \delta_{ij}.$$

Set now for $j = 1, \ldots, k$, $g_i = \sum_{l=1}^{k} a_{il} f_l$ and let $F = \{g_1, \ldots, g_k\} \cup F_2$. Then F is a proper representation of Y. □

Lemma 1.9. *Let $Y \subset l_\Phi$ or $Y \subset l_\Phi^{(m)}$ be a subspace of codimension n such that $k = \dim(r(Y^\perp)) > 0$. Suppose $F = \{f_1, \ldots, f_n\} \subset (l_\Phi)^*$ is a proper representation of Y. Let $P \in \mathcal{P}(l_\Phi, Y)$ be a projection of norm one and let $z_1, \ldots, z_n \in \ker(P)$ be given for F and P by Lemma* 1.2. *Then for $j \geqslant k+1$ and $i = 1, \ldots, n$,*

$$z_{ij} = \frac{\sum_{l=1}^{k} z_{il} \operatorname{sgn}(r(f_l)_j) \phi_l'(|r(f_l)_j|/\|y_j\|)}{\phi_j'(1/\|y_j\|)},$$

where for $j \geqslant k+1$

$$y_j = e_j - \sum_{l=1}^{k} r(f_l)_j \, e_l.$$

Proof. Let $j \geqslant k+1$ and $i \in \{1, \ldots, n\}$. Since F is a proper representation of Y, $y_j \in Y \cap h_\Phi$. By Corollary 1.6, $N(y_j)(z_i) = 0$ which gives the required equation. □

Lemma 1.10. *Let Φ, P, Y, F, n, k and $z_1, \ldots, z_n$ be as in Lemma* 1.9. *Then* $\det M \neq 0$, *where M is a $k \times k$ matrix with the ith row $m_i = (z_{i1}, \ldots, z_{ik})$, $i = 1, \ldots, k$. Moreover, for any $j \geqslant k+1$, $z_{ij} = 0$ for all $i = 1, \ldots, n$ if and only if $r(f_i)_j = 0$ for all $i = 1, \ldots, k$. Consequently,*

$$\bigcup_{j=1}^{n} \operatorname{supp}(z_j) = \bigcup_{j=1}^{k} \operatorname{supp} r(f_j).$$

Proof. If $\det M = 0$, then $m_i = \sum_{l=1, l \neq i}^{k} a_l m_l$ for some $i \in \{1, \ldots, k\}$ and $a_l \in \mathbb{R}$. By Lemma 1.9, $z_i = \sum_{l=1, l \neq i}^{n} a_l z_l$, and so

$$f_i(z_i) = \sum_{l=1, l \neq i}^{n} a_l f_i(z_l) = 0,$$

which contradicts the choice of z_i. Hence $\det M \neq 0$. Now if $z_{ij} = 0$ for some $j \geqslant k+1$ and all $i = 1, \ldots, n$, then again by Lemma 1.9,

$$u = (\operatorname{sgn}(r(f_1)_j)\phi_1'(|r(f_1)_j|/\|y_j\|), \ldots, \operatorname{sgn}(r(f_k)_j)\phi_k'(|r(f_k)_j|/\|y_j\|))$$

is a solution of the homogeneous system of linear equations given by the matrix M. Thus $u = 0$, and since ϕ_n' vanish only at zero, $r(f_j)_i = 0$ for $i = 1, \ldots, k$. On the other hand, if

$r(f_m)_j = 0$ for some $j \geqslant k+1$ and all $m = 1, \ldots, k$, then by Lemma 1.9, $z_{ij} = 0$ for all $i = 1, \ldots, n$. Moreover, since F is a proper representation of Y and $\det M \neq 0$,

$$\{1, \ldots, k\} \subset \left(\bigcup_{j=1}^{n} \operatorname{supp}(z_j) \right) \cap \left(\bigcup_{j=1}^{k} \operatorname{supp} r(f_j) \right).$$

Hence

$$\bigcup_{j=1}^{n} \operatorname{supp}(z_j) = \bigcup_{j=1}^{k} \operatorname{supp} r(f_j),$$

as required. □

Lemma 1.11. *Suppose $f_1, \ldots, f_k$ in l_{Φ^*} are such that $P_k f_1, \ldots, P_k f_k$ are linearly independent. Let $m > k$. If $v \in P_m(l_\Phi) \cap \bigcap_{i=1}^{k} \ker(f_i)$ and $\|v\| = 1$ then the elements $P_m^*(f_1), \ldots P_m^* f_k$, $N(v)$ are linearly independent.*

Proof. By the first assumption, $P_m^* f_1, \ldots, P_m^* f_k$ are linearly independent. Suppose, on a contrary, that $N(v) = P_m^* N(v) = \sum_{i=1}^{k} a_i P_m^* f_i$. Then by Theorem 1.5,

$$C_v = N(v)(v) = \sum_{i=1}^{k} a_i (P_m^* f_i)(v) = \sum_{i=1}^{k} a_i f_i(v) = 0,$$

which is a contradiction. □

Lemma 1.12. *Let $Y \subset l_\Phi$ be a subspace of codimension n and let $k = \dim(r(Y^\perp)) > 0$. Let $F = \{f_1, \ldots, f_n\}$ be a proper representation of Y such that $s(f_j) \neq 0$ for some $j \in \{1, \ldots, n\}$. Assume $P \in \mathcal{P}(l_\Phi, Y)$ and let $z_1, \ldots, z_n \in \ker P$ be given for F and P by Lemma 1.2. If z_i and $r(f_i)$ have bounded supports for $i = 1, \ldots, n$, then $\|P\| > 1$.*

Proof. Suppose that the supports of z_i and $r(f_i)$ lie in $\{1, \ldots, l\}$ for all $i = 1, \ldots, n$. It is clear that there exists $z \in (Id - P_l)(l_\Phi)$ such that $\|z\| = \rho_\Phi(z) = 1$ and $f_j(z) \neq 0$. Then z and $\sum_{i=1}^{n} f_i(z) z_i$ have disjoint supports. Hence

$$\rho_\Phi(Pz) = \rho_\Phi\left(z - \sum_{i=1}^{n} f_i(z) z_i \right) = \rho_\Phi(z) + \rho_\Phi\left(\sum_{i=1}^{n} f_i(z) z_i \right).$$

Now $f_j(z) \neq 0$ and the vectors z_i are linearly independent so that the second term on the right is non-zero. Consequently,

$$\rho_\Phi(Pz) > \rho_\Phi(z) = 1 = \|z\|,$$

which shows that $\|P\| > 1$, as required. □

2. One-complemented subspaces of finite codimension

The main results of the paper are contained in this section. Theorem 2.7 provides a complete characterization of one-complemented subspaces of finite codimension in Musielak–Orlicz spaces l_Φ for Φ satisfying condition (S). This characterization is expressed in terms of proper representations of these subspaces (see Definition 1.7 and Lemma 1.8). As a corollary, in Theorem 2.10, (see also Corollaries 2.11 and 2.12) we obtain a characterization of such subspaces in Orlicz space l_ϕ, which is an extension and completion of Theorem 7 in [19]. Furthermore, we give some corollaries on one-complemented hyperplanes in l_Φ and reproduce (Corollaries 2.18 and 2.19) the well known result on one-complemented subspaces in $l_p,\ 1 < p < \infty,\ p \neq 2$, presenting a shorter proof than that of [6].

We start with the following result, which for Orlicz spaces $l_\phi^{(m)}$ has been communicated to the authors in a slightly different form by Neubauer [18].

Theorem 2.1. *Let Φ satisfy condition (S) and let Y be a subspace of $l_\Phi^{(m)}$ of codimension $k \leqslant m-2$, which is one-complemented in $l_\Phi^{(m)}$. If $G = \{g_1, \ldots, g_k\}$ is a proper representation of Y then for any $i = 1, \ldots, k$, g_i has at most two coordinates different from zero.*

Proof. Let $Q \in \mathcal{P}(l_\Phi^{(m)}, Y)$ have norm one and let $G = \{g_1, \ldots, g_k\} \subset Y^\perp$ be a proper representation of Y. Then $Y = \bigcap_{i=1}^k \ker(g_i)$. By Lemma 1.2, there exist uniquely determined $w_1, \ldots, w_k \in \ker(P)$ satisfying $g_i(w_j) = \delta_{ij}$ and such that

$$Qx = x - \sum_{i=1}^k g_i(x) w_i, \quad x \in l_\Phi^{(m)}.$$

Assume, on a contrary, that g_j has at least three coordinates different from zero for some $j \in \{1, \ldots, k\}$. Since $l_\Phi^{(m)}$ and $l_{\Phi_\sigma}^{(m)}$ with $\Phi_\sigma = (\phi_{\sigma(n)})$ are isometric for any permutation σ of $\{1, \ldots, m\}$, and G is a proper representation of Y, by Lemma 1.3 we can assume that $g_{1p} \neq 0$ for $p = 1, k+1, k+2$.

Setting for $i \in \{1, \ldots, k\}$, $D_i = g_{1,k+1} g_{i,k+2} - g_{1,k+2} g_{i,k+1}$, define

$$A_1 = \{i = 1, \ldots, k : D_i = 0\} \quad \text{and} \quad A_2 = \{1, \ldots, k\} \setminus A_1.$$

Observe that $1 \in A_1$ and A_2 may be an empty set. Again by Lemma 1.3, without loss of generality we can assume that $A_1 = \{1, \ldots, l\}$ with $l \leqslant k$. Set as in Lemma 1.9, for $j = k+1, k+2$,

$$y_j = e_j - \sum_{p=1}^k g_{pj} e_p$$

and for $i = 1$ or each $i \in \{l+1, \ldots, k\}$ if $l < k$, we define $u_i \in Y$, by

$$u_i = \frac{g_{i,k+2} y_{k+1} - g_{i,k+1} y_{k+2}}{\| g_{i,k+2} y_{k+1} - g_{i,k+1} y_{k+2}) \|}.$$

Observe that, if $i = 1$ or $i \in \{l+1, \dots, k\}$ then

$$\|u_i\| = 1 \quad \text{and} \quad u_{ii} = 0.$$

Now, let us consider for $i = 1, \dots, k$ the following problem.

Problem (i). Minimize a function

$$x \to N(x)(w_i) = \sum_{j=1}^{m} \operatorname{sgn}(x_j)\phi_j'(|x_j|)w_{ij}$$

defined for $x \in \mathbb{R}^m$ under the conditions (C_i) given by

$$g_j(x) = \sum_{p=1}^{m} g_{jp}x_p = 0, \quad j = 1, \dots, k,$$

$$\rho_\Phi(x) = \sum_{j=1}^{m} \phi_j(|x_j|) = 1.$$

Observe that, by condition (S) and Corollary 1.6, if $x \in \mathbb{R}^m$ satisfies (C_i), then $N(x)(w_i) = 0$. This means that the function $N(\cdot)(w_i)$ has a conditional minimum at x. Let us consider at first this problem for $i = 1$. Observe that u_1 satisfies (C_i) and by Lemma 1.11 applied to $u_1, g_j, j = 1, \dots, k$, and m we get that the rank of the matrix $M(u_1) = k+1$, where for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$,

$$M(x) = \begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \cdots\cdots\cdots\cdots\cdots\cdots & & \\ g_{k1} & \cdots & g_{km} \\ \operatorname{sgn} x_1\phi_1'(|x_1|) & \cdots & \operatorname{sgn} x_m\phi_m'(|x_m|) \end{bmatrix}.$$

Since Φ satisfies condition (S), the functions $N(x)(w_i)$ and $\rho_\Phi(x)$ are continuously differentiable with respect to $x \in \mathbb{R}^m$. Hence by the Lagrange Multiplier Theorem there exist $a_p,\ p = 1, \dots, k+1$ (depending on u_1), which satisfy the following system of equations

$$E_j := \phi_j''(|u_{1j}|)w_{1j} + \sum_{p=1}^{k} a_p g_{pj} + a_{k+1}\operatorname{sgn}(u_{1j})\phi_j'(|u_{1j}|) = 0, \quad j = 1, \dots, m.$$

Since $\phi_1'(0) = \phi_1''(0) = 0$ and $g_{ij} = \delta_{ij}$ for $i, j = 1, \dots, k$, E_1 is reduced to $a_1 = 0$. Multiplying E_j by w_{1j} for $j = 1, \dots, m$ and summing up these equations we get

$$\sum_{j=1}^{m} \phi_j''(|u_{1j}|)w_{1j}^2 + \sum_{p=2}^{k} a_p\left(\sum_{j=1}^{m} g_{pj}w_{1j}\right) + a_{n+1}\sum_{j=1}^{m} \operatorname{sgn}(u_{1j})\phi_j'(|u_{1j}|)w_{1j}$$
$$= \sum_{j=1}^{m} \phi_j''(|u_{1j}|)w_{1j}^2 + \sum_{p=2}^{k} a_p g_p w_1 + a_{n+1}N(u_1)(w_1) = 0.$$

Observe that $g_p(w_1) = 0$ for $p = 2, \ldots, k$, and by the condition (S) and Corollary 1.6, $N(u_1)(w_1) = 0$. Hence the previous equality is reduced to

$$\sum_{j=1}^{m} \phi_j''(|u_{1j}|)w_{1j}^2 = 0.$$

Since ϕ_j'' vanish only at zero, $u_{1,k+1} \neq 0$ and $u_{1,k+2} \neq 0$, we get

$$w_{1,k+1} = 0 \text{ and } w_{1,k+2} = 0.$$

Moreover, if $j \in A_2$, i.e. $j \geqslant l + 1$, then $u_{1j} \neq 0$ and consequently, $w_{1j} = 0$.

Now take any $i \in A_1$. Analogously as above, minimizing the function $N(x)(w_i)$ with respect to $x \in \mathbb{R}^m$, we obtain

$$\sum_{j=1}^{m} \phi_j''(|u_{1j}|)w_{ij}^2 = 0 \tag{2.1}$$

for $i = 1, \ldots, l$. Consequently, we have that for $p \in A_1 = \{1, \ldots, l\}$ and $j \in A_2 = \{l + 1, \ldots, k\}$

$$w_{p,k+1} = w_{p,k+2} = w_{pj} = 0. \tag{2.2}$$

Let now $i \in A_2$. Minimizing the function $N(x)(w_i)$ and arguing as in the case $i = 1$, we get

$$\sum_{j=1}^{m} \phi_j''(|u_{ij}|)w_{ij}^2 = 0. \tag{2.3}$$

Consequently, in view of $u_{ij} \neq 0$ for $j \in A_1 = \{1, \ldots, l\}$ and $i \in \{l + 1, \ldots, k\}$ we have

$$w_{ij} = 0. \tag{2.4}$$

Observe that by Lemma 1.10, applied to Q, $g_1, \ldots, g_k$ and $w_1, \ldots, w_k$,

$$\det[w_{ij}]_{i,j=1,\ldots,k} \neq 0.$$

By (2.2) and (2.4), $\det[w_{ij}]_{i,j=1,\ldots,l} \neq 0$. By Lemma 1.9 applied to w_i, $i = 1, \ldots, l$, $j = k + 1$ and $j = k + 2$ we obtain

$$\phi_1'(|g_{1,k+1}|/\|y_{k+1}\|) = \phi_1'(|g_{1,k+2}|/\|y_{k+2}\|) = 0,$$

and consequently $g_{1,k+1} = g_{1,k+2} = 0$. This contradiction finishes the proof. □

In the next result we consider infinite dimensional spaces.

Theorem 2.2. *Let Φ satisfy condition (S). If Y is a one-complemented subspace of l_Φ of codimension n, then $Y^\perp$ consists of only regular functionals. Moreover, if $F = \{f_1, \ldots, f_n\}$ is a proper representation of Y then for any $i = 1, \ldots, n$, $f_i = r(f_i)$ has at most two coordinates different from zero.*

Proof. Let $P \in \mathcal{P}(l_\Phi, Y)$ with $\|P\| = 1$ and let $F = \{f_1, \dots, f_n\} \subset Y^\perp$ be a proper representation of Y. Then obviously $Y = \bigcap_{i=1}^n \ker(f_i)$. By Lemma 1.2 there exist uniquely determined $z_1, \dots, z_n \in \ker(P)$ satisfying $f_i(z_j) = \delta_{ij}$ and such that

$$Px = x - \sum_{i=1}^n f_i(x) z_i, \quad x \in l_\Phi.$$

Now we show that f_i are regular functionals, that is $s(f_i) = 0$ for $i = 1, \dots, n$. Set $k = \dim(r(Y^\perp))$. If $k = 0$, then for any $j \in \mathbb{N}$, $e_j \in Y$, since $f_i|_{h_\Phi} = s(f_i)|_{h_\Phi} = 0$ for $i = 1, \dots, n$. Consequently, by the condition (S) and Corollary 1.6, for every $j \in \mathbb{N}$ and $i = 1, \dots, n$

$$N(e_j)(z_i) = \phi_j'(1) z_{ij} = 0,$$

which yields $z_i = 0$ for all $i = 1, \dots, n$, which is impossible.

Now suppose $k > 0$. Since $\{f_1, \dots, f_n\}$ is a proper representation, $r(f_i)_j = \delta_{ij}$ for $i, j = 1, \dots k$ and $r(f_i) = 0$ for $k + 1 \leqslant i \leqslant n$, if $k < n$. First we show that for $i = 1, \dots, k$, $r(f_i)$ have at most two coordinates different from zero. Suppose this is not true. By a suitable isometry of l_Φ we can assume that $r(f_1)_{k+p} \neq 0$ for $p = 1, 2$.

We will further reduce the proof to finite-dimensional case and apply Theorem 2.1. Set for $i = 1, \dots, k$, $g_i = (r(f_i)_1, \dots, r(f_i)_{k+2})$ and $v_i = (z_{i1}, \dots, z_{i,k+2})$. Define

$$Y_1 = \bigcap_{i=1}^k \ker(g_i) \subset \mathbb{R}^{k+2} \text{ and } V_1 = \operatorname{span}[v_1, \dots, v_k] \subset \mathbb{R}^{k+2}.$$

It is clear that $\operatorname{codim}(Y_1) = k$. Moreover, by Lemma 1.10, $\det[z_{ij}]_{i,j=1,\dots,k} \neq 0$, and so $\dim(V_1) = k$. Note that

$$\mathbb{R}^{k+2} = Y_1 \oplus V_1.$$

Indeed, assume for a contrary that there exist $a_1, \dots, a_k \in \mathbb{R}$ such that $0 \neq v = \sum_{i=1}^k a_i v_i \in Y_1 \cap V_1$. Setting

$$\bar{z} = \sum_{i=1}^k a_i z_i,$$

it yields that $0 \neq w = P_{k+2}(\bar{z}) \in Y \cap h_\Phi$. Now, by Corollary 1.6,

$$0 = N(w)(\bar{z}) = \sum_{j=1}^{k+2} |w_j| \phi_j' \left(\frac{|w_j|}{\|w\|} \right)$$

and so $w_j = 0$ for $j = 1, \dots, k+2$, by condition (S). However

$$w_j = \sum_{i=1}^k a_i z_{ij} \quad \text{and} \quad \det[z_{ij}]_{i,j=1,\dots,k} \neq 0$$

by Lemma 1.10. Hence $a_i = 0$ for $i = 1, \dots, k$, which contradicts the fact that $w \neq 0$.

Now let $Q \in \mathcal{P}(l_\Phi^{k+2}, Y_1)$ be a projection with $\ker(Q) = V_1$. It is easy to see that if $0 \neq y \in Y_1$, then $\bar{y} = P_{k+2}y \in Y \cap h_\Phi$. Thus for any $0 \neq y \in Y_1$ and $w = \sum_{i=1}^k a_i v_i \in V_1$, by Corollary 1.6 we have

$$0 = N(\bar{y})(\bar{z}) = \sum_{j=1}^{k+2} \operatorname{sgn}(y_j)\phi_j'(|y_j|/\|y\|)w_j = N(y)(z).$$

Now, applying the converse statement of Corollary 1.6 to Y_1, V_1 and l_Φ^{k+2}, we get $\|Q\| = 1$. Since the codimension of Y_1 is k and $\{g_1, \ldots, g_k\}$ is a proper representation of $Y_1 \subset l_\Phi^{(k+2)}$, so by Theorem 2.1 we get

$$0 = g_{1,k+1} = r(f_1)_{k+1} = g_{1,k+2} = r(f_1)_{k+2},$$

which is a contradiction.

Consequently, by Lemma 1.10, $z_1, \ldots z_n$ have bounded supports. Since $\|P\| = 1$, by Lemma 1.12, $s(f_i) = 0$ for $i = 1, \ldots, n$. This shows that any $f \in Y^\perp$ is a regular functional.

By the first part of the proof if $\{f_1, \ldots f_n\}$ is a proper representation of Y then for any $j = 1, \ldots, n$, $f_j = r(f_j)$ has at most two coordinates different from zero. The proof is complete. $\square$

Remark 2.3. It is well known [2] that in l_∞ a one-complemented subspace of finite codimension can be an intersection of kernels of functionals with both regular and singular parts different than zero. By Theorem 2.2, this is not the case in the spaces l_Φ.

Note also that in the proof of Theorem 2.2 the assumption that Y contains at least one basic vector e_i is not needed (compare with Theorem 7 in [19]).

Lemma 2.4. *Let $Y \subset l_\Phi$ be a subspace of codimension n such that $Y^\perp$ consists of regular functionals. Let $F = \{f_1, \ldots, f_n\}$ be a proper representation of Y. Assume that there exists $l \in \mathbb{N}$ such that $\bigcup_{i=1}^n \operatorname{supp}(f_i) = \{1, \ldots, l\}$ and put $m = l+1$. Then Y is one-complemented in l_Φ if and only if $Y_1 = Y \cap l_\Phi^{(m)}$ is one-complemented in $l_\Phi^{(m)}$.*

Proof. Here we identify $l_\Phi^{(m)}$ with the subspace of those $x \in l_\Phi$ such that $x = P_m x$. Suppose that Y is one-complemented in l_Φ and let Q be a projection from l_Φ onto Y of norm one. Let $z_1, \ldots, z_n$ be given for Q and F by Lemma 1.2. In view of Lemma 1.10,

$$\bigcup_{j=1}^n \operatorname{supp}(f_j) = \bigcup_{j=1}^n \operatorname{supp}(z_j) = \{1, \ldots, l\}.$$

Hence in view of the form of Q given in Lemma 1.2, it is clear that the restriction of Q to $l_\Phi^{(m)}$ is a projection of norm one from $l_\Phi^{(m)}$ onto Y_1.

Suppose now that Q_m is a projection of norm one from $l_\Phi^{(m)}$ onto Y_1. Define

$$Qx = Q_m(P_m(x)) + (Id - P_m)x, \quad x \in l_\Phi.$$

Observe that Q is a projection from l_Φ onto Y. We shall show that $\|Q\| = 1$. Suppose on the contrary, that $\|Q\| > 1$. Then there exists $w \in l_\Phi$, such that $\|w\| = 1$ and $\|Qw\| > 1$. By definition of the Luxemburg norm on l_Φ, it is clear that

$$\rho_\Phi(Qw) > 1 \geqslant \rho_\Phi(w).$$

Hence $\rho_\Phi(Qw) = \rho_\Phi(Q_m(P_m w)) + \rho_\Phi((Id - P_m)w) > \rho_\Phi(P_m w) + \rho_\Phi((Id - P_m)w)$, and so

$$\rho_\Phi(Q_m(P_m w)) > \rho_\Phi(P_m w).$$

By Lemma 1.2 applied to Q_m and F, there exist uniquely determined $z_1 \ldots, z_n \in l_\Phi^{(m)}$ such that

$$Q_m x = x - \sum_{j=1}^{n} f_j(x) z_j, \quad x \in l_\Phi^{(m)}.$$

Moreover, by Lemma 1.10,

$$\bigcup_{j=1}^{n} \mathrm{supp}(f_j) = \bigcup_{j=1}^{n} \mathrm{supp}(z_j) = \{1, \ldots, l\}.$$

Hence for any $x \in l_\Phi$, $f_j(P_m x) = f_j(P_l x)$ for $j = 1, \ldots, n$, and so

$$Q_m(P_m x) = Q_m(P_l x) + x_m e_m.$$

Since $\rho_\Phi(P_m w) \leqslant \rho_\Phi(w) \leqslant 1$, there exists $t \geqslant 0$ such that

$$\rho_\Phi(P_l w + t e_m) = 1 = \|P_l w + t e_m\|.$$

Setting $\tilde{w} = P_l w + t e_m$, we have $\rho_\Phi(\tilde{w}) = \|\tilde{w}\| = 1$, $\tilde{w} = P_m \tilde{w} \in l_\Phi^{(m)}$ and

$$\rho_\Phi(Q_m(\tilde{w})) = \rho_\Phi(Q_m(P_l(\tilde{w})) + \tilde{w}_m e_m) = \rho_\Phi(Q_m(P_l w)) + \rho_\Phi(t e_m).$$

But $\rho_\Phi(Q_m(P_m w)) = \rho_\Phi(Q_m(P_l w)) + \rho_\Phi(w_m e_m) > \rho_\Phi(P_m w) = \rho_\Phi(P_l w) + \rho_\Phi(w_m e_m)$, and so

$$\rho_\Phi(Q_m(P_l w)) > \rho_\Phi(P_l w).$$

Hence $\rho_\Phi(Q_m(\tilde{w})) > \rho_\Phi(P_l w) + \rho_\Phi(t e_m) = \rho_\Phi(\tilde{w}) = 1$, which contradicts the assumption $\|Q_m\| = 1$. □

Theorem 2.5. *Let $Y \subset l_\Phi$ be a subspace of codimension n such that $Y^\perp$ consists of regular functionals. Let $F = \{f_1, \ldots, f_n\}$ be a proper representation of Y. Assume that for $i = 1, \ldots, n$, $f_{ij} \neq 0$ for at most one $j \geqslant n + 1$. Let for $j \in \mathbb{N}$*

$$C_j = \{i : f_{ij} \neq 0\}.$$

Then Y is one-complemented in l_Φ if and only if $Y_j = \bigcap_{i \in C_j} \ker(f_i)$ is one-complemented in l_Φ for any $j \geqslant n + 1$ such that $C_j \neq \emptyset$.

Proof. Suppose $\bigcup_{i=1}^{n} \operatorname{supp}(f_i) = \{1, \ldots, l\}$ and let $m = l + 1$. In view of the previous lemma we need to show our theorem only in the case of $l_\Phi^{(m)}$. Suppose Y is one-complemented in $l_\Phi^{(m)}$ and $J \neq \{1, \ldots, n\}$, where

$$J = \{i = 1, \ldots, n : \operatorname{supp}(f_i) = i\}.$$

Thus $f_{ij} \neq 0$ for some $j \geqslant n + 1$ and $i = 1, \ldots, n$. Fix any $j \geqslant n + 1$ with $C_j \neq \emptyset$. Let $P \in \mathcal{P}(l_\Phi^{(m)}, Y)$ be a projection of norm one. Let $z_1, \ldots, z_n \in \ker(P)$ be given for P and F by Lemma 1.2. First we claim that

$$\bigcup_{i \in C_j} \operatorname{supp}(f_i) = \bigcup_{i \in C_j} \operatorname{supp}(z_i) = C_j \cup \{j\}.$$

Indeed, let $i \in C_j$. Define $w_i \in l_\Phi^{(m)}$ by $w_{ik} = z_{ik}$ for $k \notin C_j \cup \{j\}$ and $w_{ik} = 0$ for $k \in C_j \cup \{j\}$. We shall show that $w_i = 0$. Suppose, this is not true. Since for any $k \notin C_j$, $f_k(z_i) = 0$, $w_i \in Y$. Since $z_i \in \ker(P)$, by Corollary 1.6, $N(w_i)(z_i) = 0$. By definition of w_i,

$$\begin{aligned} 0 = N(w_i)(z_i) &= \sum_{k \notin C_j} \operatorname{sgn}(w_{ik}) \phi_k'(|w_{ik}|/\|w_i\|) z_{ik}/C_{w_i} \\ &= \sum_{k \notin C_j} \operatorname{sgn}(w_{ik}) \phi_k'(|w_{ik}|/\|w_i\|) w_{ik}/C_{w_i} = \|w_i\|, \end{aligned}$$

which is a contradiction. Hence for any $i \in C_j$, $\operatorname{supp}(z_i) \subset C_j \cup \{j\}$ and consequently

$$\bigcup_{i \in C_j} \operatorname{supp}(z_i) \subset C_j \cup \{j\}.$$

We shall prove now that for any $i \in C_j$, $z_{ij} \neq 0$. Suppose, on the contrary that $z_{i_0 j} = 0$ for some $i_0 \in C_j$. Since $f_i(z_k) = \delta_{ik}$ for $i, k \in C_j$, and $\operatorname{supp}(f_i) = \{i, j\}$, we get $z_{i_0 k} = 0$ for $k \in C_j \setminus \{i_0\}$ and $z_{i_0, i_0} = 1$. Note that $y = e_j - \sum_{k=1}^{n} f_{kj} e_k \in Y$. Again, by Corollary 1.6 we get

$$0 = N(y)(z_{i_0}) = -\operatorname{sgn}(f_{i_0 j}) \phi_{i_0}'(|f_{i_0 j}|/\|z\|)$$

and consequently, $f_{i_0 j} = 0$, which is impossible.

If $i \in C_j$ and $C_j = \{i\}$, then $z_{ii} \neq 0$. Indeed, if $z_{ii} = 0$, then by Corollary 1.6 applied to y and z_i we get $-\phi_j'(1/\|y\|) z_{ij} = 0$, which is impossible since ϕ_j satisfies condition(s) and $z_{ij} \neq 0$. If $i \in C_j$ and $\operatorname{card}(C_j) > 1$, then for any $k \in C_j$ such that $k \neq i$, we have $f_i(z_k) = z_{ki} + f_{ij} z_{kj} = 0$. This shows that $z_{ki} \neq 0$, since $z_{kj} \neq 0$ and $f_{ij} \neq 0$. Consequently,

$$\bigcup_{i \in C_j} \operatorname{supp}(z_i) \supset C_j \cup \{j\}.$$

By the previous part of the proof and Lemma 1.10 we have

$$\bigcup_{i \in C_j} \operatorname{supp}(f_i) = \bigcup_{i \in C_j} \operatorname{supp}(z_i) = C_j \cup \{j\}$$

as required.

Consider now for $j \in \mathbb{N}$ the operator

$$Q_j x = x - \sum_{i \in C_j} f_i(x) z_i, \quad x \in l_\Phi^{(m)}, \tag{2.5}$$

which is a projection onto $Y_j = \bigcap_{i \in C_j} \ker(f_i)$ such that $\ker(Q_j) = V_j = \mathrm{span}[z_i : i \in C_j]$. We shall show that $\|Q_j\| = 1$. In view of Corollary 1.6, it is enough to prove that $N(y)(v) = 0$ for any $v \in V_j$ and $y \in Y_j$. Let $y = (y_1, \ldots, y_l, y_{l+1}) \in Y$ and set

$$w_j = \sum_{i \in C_j} y_i e_i + y_j e_j + t_m e_m,$$

where $t_m \geqslant 0$ is chosen so that $\|w_j\| = \|y\|$. Since for any $i \notin C_j$,

$$\mathrm{supp}(f_i) \cap \left(\bigcup_{k \in C_j} \mathrm{supp}(f_k) \right) = \emptyset,$$

$w_j \in Y$. Since $\|P\| = 1$ and $z_1, \ldots, z_n \in \ker(P)$, by Corollary 1.6, $N(w_j)(z_i) = 0$ for $i = 1, \ldots, n$. Hence

$$\sum_{k \in C_j} \mathrm{sgn}(y_k) \phi_k'(|y_k|/\|y\|) z_{ik} + \mathrm{sgn}(y_j) \phi_j'(|y_j|/\|y\|) z_{ij} = 0,$$

since $m \notin \bigcup_{i=1}^n \mathrm{supp}(z_i) = \bigcup_{i=1}^n \mathrm{supp}(f_i)$ by Lemma 1.10. Observe that, for $i \in C_j$, $N(w_j)(z_i) = N(y)(z_i)$, and thus we have that $N(y)(v) = 0$ for all $v \in V_j$. Thus, we have showed that $N(y)(v) = 0$ for all $v \in V_j$ and $y \in Y_j$. Applying now Corollary 1.6 to P_j, V_j, Y_j we have that $\|Q_j\| = 1$.

Now assume that for $j \geqslant n+1$ with $C_j \neq \emptyset$, Y_j is a one-complemented subspace of $l_\Phi^{(m)}$. We shall show that Y is one-complemented. Suppose at first that $J = \{1, \ldots, n\}$. Then

$$Y = \{y = (y_i) : y_i = 0, \ i = 1, \ldots, n\}$$

and obviously $(Id - P_n)x$ is a norm-one projection onto Y. Let now for any $j \in J_1 = \{k \geqslant n+1 : C_k \neq \emptyset\}$, Y_j be one-complemented in $l_\Phi^{(m)}$ and let $P_j \in \mathcal{P}(l_\Phi^{(m)}, Y_j)$ be a projection of norm one. Note that $F_j = \{f_i : i \in C_j\}$ is a proper representation of Y_j. Let $\{z_i^{(j)} : i \in C_j\} \subset l_\Phi^{(m)}$ be given by Lemma 1.2 for F_j and P_j. Note that, $C_i \cap C_j = \emptyset$ for $i \neq j$, $J \cap C_j = \emptyset$ for any $j \in J_1$ and $\left(\bigcup_{j \in J_1} C_j \right) \cup J = \{1, \ldots, n\}$. Define then for $i = 1, \ldots, n$,

$$w_i = \begin{cases} z_i^{(j)}, & i \in C_j, \ j \in J_1 \\ e_i, & i \in J. \end{cases}$$

The operator

$$Px = x - \sum_{i=1}^n f_i(x) w_i, \quad x \in l_\Phi^{(m)},$$

is a projection onto Y and $\ker(P) = \text{span}[w_1, \dots, w_n]$. We shall show that $\|P\| = 1$. Let $y \in Y$ and take $w_i = z_i^{(j)}$ for some $i \in C_j,\ j \in J_1$. Then $y \in Y_j$ and $z_i^{(j)} \in \ker(P_j)$. Since $\|P_j\| = 1$, by Corollary 1.6, $N(y)(z_i^{(j)}) = 0$. If $w_i = e_i$ then $i \in J$ and in that case $y_i = 0$ for any $y = (y_n) \in Y$. Thus $N(y)(w_i) = N(y)(e_i) = 0$. Therefore for any $v \in \ker(P)$ and any $y \in Y$, $N(y)(v) = 0$. Since $l_\Phi^{(m)} = h_\Phi^{(m)}$, $\|P\| = 1$ by Corollary 1.6, and the proof is complete. □

The next theorem gives a necessary and sufficient condition for the spaces Y_j considered in the proof of Theorem 2.5 to be one-complemented.

Theorem 2.6. *Let for $i = 1, \dots, n$, f_i be regular functionals on l_Φ such that $f_{ik} = \delta_{ik}$ for $i, k = 1, \dots, n$, and $f_{ij} \neq 0$ for at most one $j \geqslant n + 1$. Let for $j \in \mathbb{N}$*

$$C_j = \{i : f_{ij} \neq 0\} \quad \text{and} \quad Y_j = \bigcap_{i \in C_j} \ker(f_i).$$

Assuming that $C_j \neq \emptyset$, the subspace Y_j is one-complemented in l_Φ if and only if for all $i \in C_j$ there exist $0 \neq b_i \in \mathbb{R}$ such that for all $t \in [0, A_j]$

$$\left(\sum_{k \in C_j} \phi_k(|f_{kj} t|) + \phi_j(t)\right) b_i = \phi_i(|f_{ij} t|)/f_{ij}, \quad i \in C_j,$$

where $A_j = 1/\|y_j\|$ and $y_j = e_j - \sum_{i \in C_j} f_{ij} e_i$.

Proof. In view of Lemma 2.4 we assume that $l_\Phi = h_\Phi$ is finite dimensional. Suppose also for simplicity that $j = n + 1$, $C_j = \{1, \dots, n\}$ and $A = A_j$. Let now the equation in the hypothesis be satisfied. After differentiation we get for $i = 1, \dots, n$ and $t \in [0, 1]$

$$\begin{aligned} &\left(\sum_{k=1}^{n} |f_{k,n+1}| \phi_k' \left(\frac{|f_{k,n+1}| t}{\|y_{n+1}\|}\right) + \phi_{n+1}' \left(\frac{t}{\|y_{n+1}\|}\right)\right) b_i \\ &\quad = \text{sgn}(f_{i,n+1}) \phi_i' \left(\frac{|f_{i,n+1}| t}{\|y_{n+1}\|}\right). \end{aligned} \tag{2.6}$$

Define for $i = 1, \dots, n$

$$z_{i,n+1} = b_i,\ z_{ij} = -f_{j,n+1} b_i \quad \text{for } j \neq i,\ z_{ii} = 1 - f_{i,n+1} b_i \tag{2.7}$$

and let

$$z_i = \sum_{k=1}^{n+1} z_{ik} e_k.$$

It is clear that $f_i(z_j) = \delta_{ij}$. Hence the operator

$$Px = x - \sum_{j=1}^{n} f_j(x) z_j, \quad x \in l_\Phi,$$

is a projection from l_Φ onto Y_j. We shall show that $\|P\| = 1$. By Corollary 1.6, it is enough to prove that $N(u)(v) = 0$ for any $u \in Y_j$ and $v \in V = \text{span}[z_1, \dots, z_n]$. If $u \in Y_j$, then $u = u_{n+1}y_{n+1} + (Id - P_{n+1})y$ for some $y \in l_\Phi$. If $u_{n+1} = 0$, then $\text{supp}(u) \cap \bigcup_{i=1}^n \text{supp}(z_i) = \emptyset$ and it is clear that $N(u)(v) = 0$. If $u_{n+1} \neq 0$, then without loss of generality, we assume that $u_{n+1} = 1$. Then there exists $t \in [0, 1]$ with $1/\|u\| = t/\|y_{n+1}\|$. By (2.6)

$$\sum_{k \neq i} \text{sgn}(-f_{k,n+1})(-f_{k,n+1}) b_i \phi_k'(|f_{k,n+1}|/\|u\|) + \phi_{n+1}'(1/\|u\|) b_i$$
$$+\text{sgn}(-f_{i,n+1}) \phi_i'(|f_{i,n+1}|/\|u\|)(1 - f_{i,n+1} b_i) = 0. \tag{2.8}$$

Thus by (2.7) we obtain for $i = 1, \dots, n$

$$N(u)(z_i) = \sum_{k=1}^n z_{ik} \,\text{sgn}(-f_{k,n+1}) \phi_k'(|f_{k,n+1}|/\|y\|) + \phi_{n+1}'(1/\|y\|) z_{i,n+1} = 0. \tag{2.9}$$

Consequently, $N(u)(z_i) = 0$, for $i = 1, \dots, n$. In view of Corollary 1.6, $\|P\| = 1$.

Now assume that there is $P \in \mathcal{P}(l_\Phi, Y)$ with $\|P\| = 1$. By Lemma 1.2 applied to P, Y and $f_1, \dots, f_n$, P is determined by $z_1, \dots, z_n \in \ker(P)$, satisfying $f_i(z_j) = \delta_{ij}$. Set $b_i = z_{i,n+1}$. Since $f_{i,n+1} \neq 0$, by Lemma 1.10, $b_i \neq 0$. Observe that $z_1, \dots, z_n$ satisfy (2.7). Now, let $t \in (0, 1]$ and let $u = y_{n+1}/t$. Then $u \in Y$ and $1/\|u\| = t/\|y_{n+1}\|$. By Corollary 1.6, (2.9) is satisfied. By (2.7), (2.8) holds true, and consequently, (2.6) is satisfied. Finally integrating (2.6) over $[0, t]$, $0 \leqslant t \leqslant 1$, we obtain the required equality. □

Now we are ready to present a complete description of one-complemented subspaces of l_Φ with finite codimension. For the sake of clarity and simplicity, the description will be provided for subspaces that are expressed in terms of their proper representations. In fact the following result is a direct consequence of Theorems 2.2, 2.5 and 2.6.

Theorem 2.7. *Let Φ satisfy condition (S) and let Y be a subspace of l_Φ of codimension n. Suppose that $F = \{f_1, \dots, f_n\}$ is a proper representation of Y. Then Y is one-complemented if and only if the following conditions hold.*

(a) *The subspace $Y^\perp$ consists of only regular functionals. In particular, $f_i = r(f_i)$ for every $i = 1, \dots, n$.*
(b) *For any $i = 1, \dots, n$, f_i has at most two non-zero coordinates and $f_{ij} = \delta_{ij}$ for $i, j = 1, \dots, n$.*
(c) *Let for $j \in \mathbb{N}$*

$$C_j = \{i : f_{ij} \neq 0\}.$$

Then for any $j \geqslant n+1$ with $C_j \neq \emptyset$ and for any $i \in C_j$ there exist $b_{ij} > 0$ such that for all $t \in [0, A_j]$

$$\left(\sum_{k \in C_j} \phi_k(|f_{kj} t|) + \phi_j(t) \right) b_{ij} = \phi_i(|f_{ij} t|)/|f_{ij}|,$$

where $A_j = 1/\|y_j\|$ and $y_j = e_j - \sum_{i \in C_j} f_{ij} e_i$.

As a corollary of the above theorem we will obtain a characterization of one-complemented subspaces of finite codimension in Orlicz spaces. In this case the characterization achieves a simpler form. But at first we will need two technical lemmas. Recall that two Orlicz functions $\phi_1, \phi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are equivalent at zero whenever there exist $u_0 > 0$ and positive constants $M_i, m_i,\ i = 1, 2$, such that for $t \in [0, u_0]$

$$m_1\phi_1(m_2t) \leqslant \phi_2(t) \leqslant M_1\phi_1(M_2t).$$

The following two lemmas for a single Orlicz function are well known (cf. [14,19]).

Lemma 2.8. *Let $\phi_i,\ i = 1, 2$, be Orlicz functions. Suppose that there exist $A, b, b_1 > 0$ such that for all $u \in [0, A]$ and $a \in (0, 1)$*

$$\phi_1(u) = b\phi_2(u) \quad \textit{and} \quad \phi_1(au) = b_1 a\phi_2(u). \tag{2.10}$$

Let $p = \log_a((b_1/b)a)$. Then $\phi_i(u)$ and u^p are equivalent at zero, that is there exist positive constants m, M such that

$$mu^p \leqslant \phi_i(u) \leqslant Mu^p \tag{2.11}$$

for all $u \in [0, A]$ and $i = 1, 2$.

Proof. Observe that by (2.10), $\phi_1(au) = ac\phi_1(u)$ for $u \in [0, A]$, where $c = b_1/b$. Let m_0 be the smallest natural number satisfying $a^{m_0} \leqslant A$. If $a^{m_0} < u \leqslant A$, then

$$(\phi_1(a^{m_0})/A^p)u^p \leqslant \phi_1(u) \leqslant \phi_1(A) \leqslant (\phi_1(A)/a^{m_0 p})u^p.$$

If $a^{m+1} < u \leqslant a^m$ for some $m \geqslant m_0$ then by (2.10) and definition of p

$$\begin{aligned} \phi_1(u) &\leqslant \phi_1(a^m) = ca\phi_1(a^{m-1}) = \cdots = (ca)^{m-m_0}\phi_1(a^{m_0}) \\ &= \phi_1(a^{m_0})a^{(m+1)p}/(ca)^{m_0+1} \leqslant (\phi_1(a^{m_0})/(ca)^{m_0+1})u^p. \end{aligned}$$

Analogously,

$$\begin{aligned} \phi_1(u) &\geqslant \phi_1(a^{m+1}) = ca\phi_1(a^m) = \cdots = (ca)^{m+1-m_0}\phi_1(a^{m_0}) \\ &= (\phi_1(a^{m_0})a^{mp})/(ca)^{m_0-1} \geqslant (\phi_1(a^{m_0})/(ca)^{m_0-1})u^p. \end{aligned}$$

Setting

$$m = \min\{\phi_1(a^{m_0})/A^p, \phi_1(a^{m_0})/(ca)^{m_0-1}\}$$

and

$$M = \max\{\phi_1(A)/a^{m_0 p}, \phi_1(a^{m_0})/(ca)^{m_0+1}\},$$

by the previous inequalities we obtain for all $u \in [0, A]$, $mu^p \leqslant \phi_1(u) \leqslant Mu^p$. In view of (2.10), we easily get (2.11) and complete the proof. □

Lemma 2.9. *Let ϕ_1, ϕ_2 be Orlicz functions. Suppose $a, b \in (0, 1), a < b$, are such that there is no $w > 0$ with the property that $a, b \in \{w^k : k \in \mathbb{Z}\}$. Moreover, let for some $A > 0$ there exist $c, c_a, c_b > 0$ such that for any $u \in [0, A]$*

$$\phi_1(u) = c\phi_2(u), \phi_1(au) = c_a a\phi_2(u), \phi_1(bu) = c_b b\phi_2(u).$$

Then there exist $K_i > 0,\ i = 1, 2$, *and* $1 \leqslant p < \infty$, *such that*

$$\phi_i(t) = K_i t^p, \quad t \in [0, A],\ i = 1, 2.$$

Proof. It is enough to show our assertion only for ϕ_1. Let $h : (-\infty, \ln \mathrm{A}] \to \mathbb{R}$ be given by

$$h(t) = (\ln \phi_1(e^t))'.$$

By the equality in the assumption, for $t \in (-\infty, \ln A]$,

$$\phi_1(e^t b) = (c_b b/c)\phi_1(e^t)$$

and consequently,

$$\ln \phi_1(e^{t+\ln b}) = \ln(c_b b/c) + \ln \phi_1(e^t).$$

After differentiation of both sides of the above equality we get

$$h(t + \ln b) = h(t) \quad \text{for } t \in (-\infty, \ln A]$$

and analogously,

$$h(t + \ln a) = h(t) \quad \text{for } t \in (-\infty, \ln A].$$

We claim that h is a constant function on $(-\infty, \ln A]$. Indeed, if h is not constant, then in view of the above equalities it must be periodic on $(-\infty, \ln A]$, and thus $\ln b = k \ln a$ for some integer $k \neq 1$. Hence $b = a^k$, which contradicts the assumptions on a and b. So h is a constant function on $(-\infty, \ln A]$ and consequently $\ln \phi_1(e^t) = pt + D$ for some constants p, D. Hence $\phi_1(u) = e^D u^p$ on $[0, A]$ with $p \geqslant 1$, which proves our assertion. □

The next three results complete the well known characterization obtained in [19], Theorem 7, as well as extending it to a nonseparable case. The first theorem provides necessary and sufficient conditions for a subspace of an Orlicz space with finite codimension to be one-complemented.

Theorem 2.10. *Let* ϕ *satisfy condition* (S) *and* $Y \subset l_\phi$ *be a subspace of codimension* n. *Let* $F = \{f_1, \ldots, f_n\}$ *be a proper representation of* Y. *Then* Y *is one-complemented in* l_ϕ *if and only if the following conditions hold.*

(a) *The subspace* $Y^\perp$ *consists of regular functionals. In particular* $r(f_i) = f_i$ *for* $i = 1, \ldots, n$.

(b) *For any* $i = 1, \ldots, n$, f_i *has at most two coordinates different from zero and* $f_{ij} = \delta_{ij}$ *for* $i, j = 1, \ldots, n$.

(d) *Let for* $j \in \mathbb{N}$

$$C_j = \{i : f_{ij} \neq 0\}.$$

Then for any $j \geqslant n + 1$ *with* $C_j \neq \emptyset$ *and for any* $i \in C_j$ *there exist* $c_{ij} > 0$ *such that for all* $t \in [0, A_j]$

$$\phi(|f_{ij}|t) = c_{ij}|f_{ij}|\phi(t),$$

where $A_j = 1/\|y_j\|$ *and* $y_j = e_j - \sum_{i \in C_j} f_{ij} e_i$.

Proof. In view of Theorem 2.7 we only need to show that condition (d) is equivalent to condition (c) in the case of Orlicz spaces. Assume first that condition (c) of Theorem 2.7 holds true. Since $\phi_n = \phi$ for all $n \in \mathbb{N}$, comparing the right-hand sides of the equalities in condition (c) we get

$$\phi(|f_{ij}/f_{kj}|u) = b_{ij}|f_{ij}/b_{kj}f_{kj}|\phi(u)$$

for all $i \neq k \in C_j$ and $u \leqslant |f_{kj}|A_j$. Thus for any $k_0 \in C_j$ and any $u \leqslant |f_{k_0 j}|A_j$,

$$\sum_{k \in C_j, k \neq k_0} |b_{kj}f_{kj}/b_{k_0 j}f_{k_0 j}|\phi(u) + \phi(u) + \phi(u/|f_{k_0 j}|) = \phi(u)/|b_{k_0 j}f_{k_0 j}|,$$

or equivalently $\phi(|f_{k_0 j}|t) = c_{k_0 j}|f_{k_0 j}|\phi(t)$ for all $t \in [0, A_j]$ and some $c_{k_0 j} > 0$. Hence (d) holds.

Now assume that condition (d) holds. Then for $t \in [0, A_j]$

$$\sum_{k \in C_j} \phi(|f_{kj}|t) = b_{ij}\phi(|f_{ij}|t)/|f_{ij}|,$$

where $b_{ij} = \left(1 + \sum_{k \in C_j} c_{kj}|f_{kj}|\right)/c_{ij}$. The proof is complete. □

Notice that the numbers f_{ij} in condition (d) above are multipliers of ϕ in a neighborhood of zero (cf. [14]). Moreover, it is clear that if f_{ij} are either zero or one, then this condition is always satisfied. In view of Lemma 2.8 it appears that it is the only possibility in the case when ϕ is not equivalent to a power function. Thus we can state the following corollary.

Corollary 2.11. *Let ϕ and Y satisfy the assumptions of Theorem* 2.10. *If ϕ is not equivalent to a power function at zero, then Y is one-complemented in l_ϕ if and only if conditions* (a), (b) *and* (d′) *are satisfied, where*
(d′) $|f_{ij}| \in \{0, 1\}$ *for any* $i = 1, \ldots, n$ *and* $j \in \mathbb{N}$.

By Lemma 2.9 and Theorem 2.10 we also obtain the following observation.

Corollary 2.12. *Let ϕ and Y satisfy the assumptions of Theorem* 2.10. *Suppose also that ϕ is an Orlicz function which does not coincide to a power function in a neighborhood of zero. Then if Y is a one-complemented subspace in l_ϕ, then conditions* (a), (b) *and* (d″) *must be satisfied, where*
(d″) *There exists $w > 0$ such that*

$$|f_{ij}| \in \{w^k : k \in \mathbb{Z}\}.$$

Hyperplanes, the kernels of functionals, are of special importance. The next few corollaries provide characterizations of some one-complemented hyperplanes.

Corollary 2.13. *Let $f = e_1 + f_2 e_2$, where $f_2 \neq 0$ and let $A = 1/\|(-f_2)e_1 + e_2\|$. Then $Y = \ker(f)$ is one-complemented in l_Φ if and only if there exists $c \neq 0$ such that for every $u \in [0, A]$*

$$\phi_1(|f_2|u) = cf_2\phi_2(u).$$

Proof. By Theorem 2.6, Y is one-complemented if and only if there exists $b \neq 0$, such that

$$(\phi_1(|f_2|u) + \phi_2(u))b = \phi_1(|f_2|u)/f_2$$

for $0 \leqslant u \leqslant A$. Hence

$$\phi_1(|f_2|u)(1 - f_2 b) = bf_2\phi_2(u).$$

Consequently, $1 - f_2 b \neq 0$ and $c = b/(1 - f_2 b)$ satisfies the required equation. □

Corollary 2.14. *Let Φ satisfy condition (S) and suppose that for any $j \neq k$, ϕ_j and ϕ_k are not equivalent at zero. Then given $0 \neq f \in (l_\Phi)^*$, $Y = \ker(f)$ is one-complemented in l_Φ if and only if f is a regular functional having exactly one coordinate different from zero.*

Proof. If $f = (f_n)$ is a regular functional with exactly one coordinate different than zero then clearly $Y = \ker(f)$ is one-complemented. Now, if $Y = \ker(f)$ is one-complemented then by Theorem 2.2, $f = r(f)$ and f has at most two coordinates different from zero. Assuming that $f = e_1 + f_2 e_2$ and $f_2 \neq 0$, by Corollary 2.13, ϕ_1 must be equivalent to ϕ_2, which contradicts the assumptions. □

Example 2.15. Let l_Φ be a Nakano space, that is $\phi_j(t) = t^{p_j}$, $1 \leqslant p_j < \infty$, for $j \in \mathbb{N}$. As a direct consequence of Corollary 2.14, if $p_j > 2$ and $p_j \neq p_l$ for $j \neq l$, then $\ker(f)$ is one-complemented in l_Φ if and only if $f_j \neq 0$ for exactly one $j \in \mathbb{N}$.

Corollary 2.16. *Suppose $\phi(t) = Dt^p$ for some $D > 0$, $p \geqslant 1$ on $[0, a]$, where $[0, a]$ is the largest interval having this property. If $a < \phi^{-1}(1/2) = 1/\|e_1 + e_2\|$, then for any $0 < f < 1$, $\ker(e_1 + fe_2)$ is not one-complemented in Orlicz space l_ϕ.*

Proof. Suppose $\ker(e_1 + fe_2)$ is one-complemented in l_ϕ for some $0 < f < 1$. By Corollary 2.13, there exists $c > 0$ such that

$$\phi(fu) = cf\phi(u), \quad 0 < u \leqslant 1/\|fe_1 + e_2\|.$$

Then by the same argument as in Lemma 2.9, the function $h(t) = (\ln \phi(e^t))'$ is either constant or periodic on $(-\infty, \ln A]$, where $A = 1/\|e_1 + e_2\|$. Since $\phi(t) = Dt^p$ on $[0, a]$, h is constant on $(-\infty, \ln a]$ and consequently, $\phi(t) = Dt^p$ on $[0, A]$, which is a contradiction. □

Corollary 2.17. *Suppose $\phi(t) = Dt^p$ on $[0, a]$, where $a \geqslant \phi^{-1}(1/2) = 1/\|e_1 + e_2\|$ and let $[0, a]$ be the largest interval having this property. Suppose $0 < f \leqslant 1$. Then $\ker(e_1 + fe_2)$ is one-complemented in l_ϕ if and only if $A_f = 1/\|e_1 + fe_2\| \leqslant a$.*

Proof. Suppose $\ker(e_1 + fe_2)$ is one-complemented in l_ϕ and $A_f > a$. Reasoning as in Corollary 2.16, we get $\phi(t) = Dt^p$ on $[0, A_f]$, which is a contradiction. Now suppose that $A_f \leqslant a$ and take any $u \in [0, A_f]$. Obviously

$$\phi(fu) = f^{p-1} f\phi(u),$$

since $0 \leqslant u \leqslant A_f \leqslant a$. Thus by Corollary 2.13, $\ker(e_1 + f e_2)$ is one-complemented in l_ϕ. □

Finally, we apply Theorem 2.2 and Corollary 2.13 to provide a different proof of a well known result [6] on one-complemented subspaces of finite codimension in l_p $(1 < p < \infty)$, $p \neq 2$. First, we consider the case of hyperplanes.

Corollary 2.18. *Let $1 < p < \infty$, $p \neq 2$. Then given $f \in l_{p'}$, $Y = \ker(f)$ is one-complemented in l_p if and only if f has at most two coordinates different from zero.*

Proof. Let *Y* be one-complemented and first suppose that $p > 2$. By Theorem 2.2, *f* has at most two coordinates different than zero. Now suppose $1 < p < 2$, and let $P \in \mathcal{P}(l_p, Y)$ be a projection of norm-one. By Lemma 1.2, applied to *P* and *f* there exists $z \in \ker(P)$ such that $f(z) = 1$ and for $x \in l_p$,

$$Px = x - f(x)z.$$

Moreover, by Lemma 1.10, $\mathrm{supp}(z) = \mathrm{supp}(f)$. Note, that the adjoint operator $P^*x = x - z(x)f$, $x \in l_{p'}$, is a projection of norm one from $l_{p'}$ onto $\ker(z)$. Since $p' > 2$, by the previous part of the proof, z has at most two coordinates different from zero, which shows that *f* has also at most two coordinates different from zero.

Suppose now that *f* has exactly two coordinates different from zero. Without loss of generality we assume that $f_1 = 1$ and $f_2 > 0$, and put $c = f_2^{p-1}$. Since the function $t \to t^p$ is *p*-homogeneous, the equation in Corollary 2.13 is satisfied for any $u \geqslant 0$ and thus $\ker(f)$ is one-complemented in l_p. □

Corollary 2.19. *Suppose $Y \subset l_p$ is a subspace of finite codimension, where $1 < p < \infty$, $p \neq 2$. Let $F = \{f_1, \ldots, f_n\}$ be a proper representation of Y. Then Y is one-complemented in l_p if and only if for every $i = 1, \ldots, n$ there is at most one $j \geqslant n+1$ such that $f_{ij} \neq 0$.*

Proof. Suppose *Y* is one-complemented in l_p. If $p > 2$, then by Theorem 2.2, for any $i = 1, \ldots, n$ there is at most one $j \geqslant n+1$ with $f_{ij} \neq 0$. Now, let $1 < p < 2$ and suppose for a contrary that there exists $j \in \{1, \ldots, n\}$ and $n+1 \leqslant l < k$ such that $f_{il} \neq 0$ and $f_{ik} \neq 0$. Without loss of generality we assume that $i = 1, k = n+1, l = n+2$ and $f_{1,n+1}, f_{1,n+2} > 0$. Let $z_1, \ldots z_n \in l_p$ be given for *P* and *F* by Lemma 1.2. Setting

$$w = e_{n+1} - \sum_{j=1}^{n} f_{j,n+1} e_j \quad \text{and} \quad z = e_{n+1} - \sum_{j=1}^{n} f_{j,n+2} e_j,$$

it is clear that $w, z, tz, w + tz \in \ker(P)$, $t \geqslant 0$. By Corollary 1.6, applied to w, tz, $w + tz$, respectively, we obtain the following equations for $i = 1, \ldots, n$

$$z_{i,n+1} = \sum_{k=1}^{n} z_{ik} f_{k,n+1}^{p-1},$$

$$t^{p-1}z_{i,n+2} = \sum_{k=1}^{n} z_{ik}(tf_{k,n+2})^{p-1},$$
$$z_{i,n+1} + t^{p-1}z_{i,n+2} = \sum_{k=1}^{n} z_{ik}(f_{k,n+1} + tf_{k,n+2})^{p-1}.$$

By Lemma 1.10, the matrix M with rows $m_i = (z_{i1}, \ldots, z_{in})$ is invertible, so the system of equations

$$\sum_{k=1}^{n} [(f_{k,n+1} + tf_{k,n+2})^{p-1} - (f_{k,n+1})^{p-1} - (tf_{k,n+1})^{p-1}]z_{ik} = 0$$

has only a trivial solution. This in particular yields the following equation.

$$(f_{1,n+1}/2)^{p-1} + (tf_{1,n+2}/2)^{p-1} = ((f_{1,n+1} + tf_{1,n+2})/2)^{p-1} \quad \text{for } t \geqslant 0.$$

Since $0 < p - 1 < 1$, by strict concavity of the function $s \to s^{p-1}$, we get $f_{1,n+1} = 0$ or $f_{1,n+2} = 0$, which is a contradiction.

Now suppose that $Y = \bigcap_{j=1}^{n} \ker(f_j)$, where $f_j \in l_{p'}$, $f_{ij} = \delta_{ij}$ for $i, j = 1, \ldots, n$. If $f_j = e_j$ for $j = 1, \ldots, n$, then $Id - P_n$ is a projection onto Y of norm one. So assume $f_{ij} \neq 0$ for at most one $j \geqslant n+1$. We need to show that Y is one-complemented. By Theorem 2.5, we can reduce the problem to the case of subspaces considered in Theorem 2.6. Thus we can assume that for each $i = 1, \ldots, n$, $f_{i,n+1} \neq 0$. Let for $i = 1, \ldots, n$, $z_i = \sum_{j=1}^{n+1} (z_i)_j e_j$ be a solution of the system (E_i) of linear equations given by:

$$z_{ii} + f_{i,n+1}z_{i,n+1} = 1,$$
$$z_{ij} + f_{j,n+1}z_{i,n+1} = 0 \quad \text{for } j \neq i,$$
$$\left(\sum_{j=1}^{n} -\operatorname{sgn}(f_{j,n+1})|f_{j,n+1}|^{p-1}z_{ij}\right) + z_{i,n+1} = 0. \tag{2.12}$$

By Lemma 1.11, applied to $v = \sum_{j=1}^{n} (-f_{j,n+1})e_j + e_{n+1} \in Y$, there exists exactly one solution of system (E_i). Moreover, $z_{i,n+1} \neq 0$ for $i = 1, \ldots, n$, by the assumption $f_{i,n+1} \neq 0$. Put for $i = 1, \ldots, n$, $b_i = z_{i,n+1}$. Since for $i = 1, \ldots, n$, $z_{ii} = 1 - f_{i,n+1}b_i$ and $z_{ij} = -f_{j,n+1}b_i$ for $j \neq i$, by (2.12), we get

$$\left(\sum_{j=1}^{n} |f_{j,n+1}|^p + 1\right)b_i = |f_{i,n+1}|^p / f_{i,n+1}.$$

Since the function $t \to t^p$ is p-homogeneous, by Theorem 2.6, Y is one-complemented in l_p. In fact, the projection with kernel $V = \text{span}[z_1, \ldots, z_n]$, where $z_1, \ldots, z_n$ is a solution of the system (E_i) has norm one. □

3. Intersections of one-complemented hyperplanes

From the results of [6,7] it follows that a subspace $Y \subset l_p$, $1 \leqslant p < \infty$, of finite codimension is one-complemented if and only if Y can be represented as an intersection of one-complemented hyperplanes. As we will see below, in Musielak–Orlicz spaces satisfying condition (S) only the sufficiency part of this statement holds true. However, in general the intersection of one-complemented hyperplanes need not be one-complemented. Indeed, let $X = l_\infty^{(4)}$, $f_1 = (1, 0, 0, 0)$, $f_2 = (1/2, 1/6, 1/6, 1/6)$ and let $Y = \ker(f_1) \cap \ker(f_2)$. Recall [4] that for any $f = (f_1, \ldots, f_n) \in l_1^{(n)}$ with $\|f\|_1 = 1$, $Y = \ker(f)$ is one-complemented in $l_\infty^{(n)}$ if and only if $|f_i| \geqslant 1/2$ for some $i = 1, \ldots, n$. Moreover, $Px = x - f(x)z$, $x \in l_\infty^{(n)}$, with $z = (1/f_i)e_i$, is a one norm projection onto Y. Hence $\ker(f_i)$, $i = 1, 2$, are one-complemented in X. We also observe that $Y = \ker(f_1) \cap \ker(f_3)$, where $f_3 = (0, 1/3, 1/3, 1/3)$. By [4], $\ker(1/3, 1/3, 1/3)$ is not one-complemented in $l_\infty^{(3)}$. We conclude that Y is also not one-complemented in X, applying the following result.

Let $f \in l_1^{(n-1)} \setminus \{0\}$ and let $Y = \ker(f)$. Let $h \in l_1^{(n)}$, $\|h\|_1 = 1$, $|h_1| \geqslant 1/2$. Set $Y_1 = \ker(0, f) \cap \ker(h) \subset l_\infty^{(n)}$. Then the norm of minimal projection from $l_\infty^{(n-1)}$ onto Y is equal to the norm of minimal projection from $l_\infty^{(n)}$ onto Y^1 [15, Theorem 1.11].

Observe here that $z_1 = e_1 = z_2/2$, where for $i = 1, 2$, z_i determine the minimal projections onto $\ker(f_i)$ (see Lemma 1.2). The vectors z_i are linearly dependent, and as we will see below it is a different situation than in l_Φ.

Theorem 3.1. *Let Φ satisfy condition (S). Suppose $Y \subset l_\Phi$ is a subspace of codimension n and $Y = \bigcap_{i=1}^n \ker(f_i)$, where $f_i \in (l_\Phi)^*$. If $\ker(f_i)$ is one-complemented in l_Φ for each $i = 1, \ldots, n$, then Y is one-complemented in l_Φ.*

Proof. Let $\ker(f_i)$ be one-complemented in l_Φ for each $i = 1, \ldots, n$. Notice that if f is a functional on l_Φ, then multiplying f by a suitable number $\neq 0$ we obtain a proper representation of a subspace $\ker(f)$. Thus we can apply Theorem 2.2 to each subspace $\ker(f_i)$. Hence each f_i must be a regular functional, that is $r(f_i) = f_i$, such that it has at most two coordinates different from zero. By Lemma 1.2, for each $i = 1, \ldots, n$ there exists $z_i \in l_\Phi$ such that $f_i(z_i) = 1$ and $Q_i x = x - f_i(x)z_i$, $x \in l_\Phi$ is a norm one projection onto $\ker(f_i)$. Since $f_i = r(f_i)$ for $i = 1, \ldots, n$ and $\operatorname{codim}(Y) = n$, $f_1, \ldots, f_n$ are linearly independent. Set

$$K = \bigcup_{j=1}^{n} \operatorname{supp}(f_j)$$

and $m = 1 + \operatorname{card} K$. By a suitable permutation of $\mathbb{N}$, which induces an isometry of l_Φ and by Lemma 1.3, without loss of generality we can assume that $K = \{1, \ldots, m-1\}$. Let for $i = 1, \ldots, n$

$$g_i = (f_{i1}, \ldots, f_{im}) \quad \text{and} \quad w_i = (z_{i1}, \ldots, z_{im})$$

and $Y_m = \bigcap_{l=1}^n \ker(g_i)$. It is clear that $Y_m = Y \cap l_\Phi^{(m)}$. Following the proof of Lemma 2.4, it is easy to see that we need only to show that Y_m is one-complemented in $l_\Phi^{(m)}$. Let

$$D = \{j \in \bigcup_{i=1}^{n} \operatorname{supp}(g_i) : y_j = 0 \text{ for any } y \in Y_m\}.$$

It is not difficult to observe that $\operatorname{card} D \leqslant n$. Further we shall consider two cases.

Case I: Suppose $D = \emptyset$. We claim that in this case for any $k \in \{1, \ldots, n\}$ and $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$

$$\mathrm{card} \bigcup_{i=1}^{k} \mathrm{supp}(g_{j_i}) \geqslant k + 1.$$

Assume on a contrary, that the above inequality does not hold for some $k \in \{1, \ldots, n\}$ and $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$. Without loss of generality, we can assume that $j_i = i$ for $i = 1, \ldots, k$. Since $g_1, \ldots, g_k$ are linearly independent,

$$\mathrm{card} \bigcup_{j=1}^{k} \mathrm{supp}(g_j) = \{i_1, \ldots, i_k\}) = k$$

and $\det[g_{ji_l}]_{j,l=1,\ldots,k} \neq 0$. Then for any $y \in Y_m$ and any $j = 1, \ldots, k$, the only solutions of

$$0 = g_j(y) = \sum_{l=1}^{k} y_{i_l} g_{ji_l},$$

are $y_{i_l} = 0$ for all $l = 1, \ldots, k$, which is a contradiction. Thus we proved the claim and in particular, for any $j = 1, \ldots, n$, $\mathrm{card}\,\mathrm{supp}(g_j) = 2$.

Now we shall show by induction that the elements w_j, $j = 1, \ldots, n$ are linearly independent. For $n = 1$ this is obviously true. Now assume that any $n - 1$ element subset of $\{w_1, \ldots, w_n\}$ is linearly independent. Set for any $j \in K$,

$$C_j = \{i = 1, \ldots, n : g_{ij} \neq 0\}.$$

Then there exists $j \in K$ with $\mathrm{card}(C_j) = 1$. In fact, if this is not true, then without loss of generality, we can assume

$$2 \leqslant \mathrm{card}\, C_j \leqslant \mathrm{card}\, C_{j+1}$$

for any $j = 1, \ldots, m - 2$. Observe that for $j > 1$, $\mathrm{card}\, C_j \backslash C_1 \geqslant 1$. If $\mathrm{card}\, C_j \backslash C_1 = 0$ for some $j > 1$ then $C_1 \cup C_j = C_1$ and so

$$\mathrm{card}(\mathrm{supp}(g_i) \cup \mathrm{supp}(g_l) = \{1, j\}) = 2$$

for some $i \neq l \in \{1, \ldots, n\}$, $i \neq l$, which contradicts the previous part of the proof. Hence, in particular, $\mathrm{card}\, C_1 \cup C_2 \geqslant 3$. Repeating this procedure n times we get

$$\mathrm{card} \bigcup_{j=1}^{n} C_j \geqslant n + 1$$

which contradicts the fact that

$$\bigcup_{j=1}^{n} C_j \subset \{1, \ldots, n\}.$$

Consequently, there exists $j \in \{1, \ldots, k\}$ such that $C_j = \{i_0\}$. Now applying the induction hypothesis we get that $w_1, \ldots, w_{i_0-1}, w_{i_0+1}, \ldots w_n$ are linearly independent. Observe that by Lemma 1.10 applied to $\ker(g_i)$ and w_i,

$$j \notin \bigcup_{i \neq i_0} \operatorname{supp}(g_i) = \bigcup_{i \neq i_0} \operatorname{supp}(w_i).$$

Since $j \in \operatorname{supp}(g_{i_0}) = \operatorname{supp}(w_{i_0})$, $w_{i_0} \notin \operatorname{span}[w_i : i \neq i_0]$, which shows that $w_1, \ldots, w_n$ are linearly independent.

In order to show that Y_m is one-complemented, set $V = \operatorname{span}[w_1, \ldots, w_n]$. Since $g_i(w_i) = f_i(z_i) = 1$, for any $i = 1, \ldots, n$, $V \cap Y_m = \{0\}$. Thus $l_\Phi^{(m)} = V \oplus Y_m$ and the natural projection Q onto Y_m is bounded. We will show that $\|Q\| = 1$. Take any $y \in Y_m$ and $i \in \{1, \ldots, n\}$. Since $y \in \ker(g_i) \subset h_\Phi$, applying Corollary 1.6 to $Q_i|_{\mathbb{R}^m}$ and $\ker(g_i)$, we get $N(y)(w_i) = 0$. Consequently $N(y)(v) = 0$ for any $v \in V$. By Corollary 1.6, $\|Q\| = 1$. Thus Y_m is one-complemented in $l_\Phi^{(m)}$, and so Y is one-complemented in l_Φ.

Case II: Let $D \neq \emptyset$. If card $D = n$, then

$$Y_m = \{y \in l_\Phi^{(m)} : y_j = 0 \text{ for } j \in D\}$$

and obviously it is a one-complemented subspace of $l_\Phi^{(m)}$. If $\operatorname{card}(D) < n$, set

$$I = \{i = 1, \ldots, n : \operatorname{supp}(g_i) \cap D = \emptyset\}.$$

Note that, if $\operatorname{supp}(g_i) \cap D \neq \emptyset$, then $\operatorname{supp}(g_i) \subset D$. Hence $I \neq \emptyset$. Applying the first part of the proof to $Z = \bigcap_{i \in I} \ker(g_i)$, we can find $Q \in \mathcal{P}(l_\Phi^{(m)}, Z)$ of norm one. Set for $x = (x_1, \ldots, x_m) \in l_\Phi^{(m)}$, $Rx = ((Rx)_1, \ldots, (Rx)_m)$, where $(Rx)_i = 0$ if $i \in D$ and $(Rx)_i = x_i$ in the opposite case. Finally it is easy to see that $P = R \circ Q$ is a norm one projection of $l_\Phi^{(m)}$ onto Y_m and the proof is complete. □

The last two results in this section show that in some Musielak–Orlicz and Orlicz spaces there exist one-complemented subspaces of finite codimension which cannot be represented as an intersection of one-complemented hyperplanes.

Theorem 3.2. *Let Φ satisfy condition (S). Assume also that there exists $0 < u_0 < 1/2$ such that*

$$\phi_j(u) = \phi_1(u) \tag{3.1}$$

for all $j \in \mathbb{N}$ and $u \in [0, u_0]$. Assume additionally that for any $j, k \in \mathbb{N}$, $j \neq k$, there is $t \in (u_0, 1/2)$ such that

$$\phi_j(t) \neq \phi_k(t). \tag{3.2}$$

Then there exists a one-complemented subspace $Y \subset l_\Phi$ of finite codimension which cannot be represented as an intersection of one-complemented hyperplanes.

Proof. Put $a_n = \| \sum_{j=1}^n e_j \|$. Observe that $a_n \to \infty$; if not, then for every $n \in \mathbb{N}$, $1/a_n > a > 0$ for some $a \in [0, u_0]$, and then for every $n \in \mathbb{N}$

$$1 = \sum_{j=1}^{n} \phi_j(1/a_n) > \sum_{j=1}^{n} \phi_j(a) = n\phi_1(a),$$

which is a contradiction. Now, fix $n \in \mathbb{N}$ with $1/a_n < u_0$. Put for $i = 1, \ldots, n$, $b_i = 1/(n+1)$, $f_i = e_i + e_{n+1}$ and let $Y = \bigcap_{i=1}^n \ker(f_i)$. Observe that for any $t \in [0, u_0]$, f_i and b_i satisfy the equality in Theorem 2.6. Since

$$1/\|e_{n+1} - \sum_{j=1}^{n} e_j\| \leqslant 1/\|a_n\| < u_0,$$

by Theorem 2.6, Y is one-complemented in l_Φ.

Now we shall show that Y cannot be represented as an intersection of one-complemented hyperplanes. First we claim that for any $i = 1, \ldots, n$, $\ker(f_i)$ is not one-complemented in l_Φ. Since for any $i = 1, \ldots, n$, $\| - e_i + e_{n+1}\| \leqslant 2$, by Corollary 2.13

$$\phi_i(t) = c\phi_{n+1}(t)$$

for $t \in [0, 1/2]$, if $\ker(f_i)$ is one-complemented. But it follows that $c = 1$ by (3.1). However, this is impossible in view of (3.2), which proves our claim.

Now suppose that $Y = \bigcap_{i=1}^n \ker(g_i)$, where $g_i \in (l_\Phi)^*$ and $\ker(g_i)$ are one-complemented for $i = 1, \ldots, n$. Then for each $i = 1, \ldots, n$, $g_i = \sum_{j=1}^n a_{ij} f_j$ for some $a_{ij} \in \mathbb{R}$. By Theorem 2.2, $g_i = r(g_i) \in l_{\Phi^*}$ and g_i have at most two coordinates different from zero. This means that $g_i = -be_k + be_l$ for some $k, l \in \{1, \ldots, n\}$, $k \neq l$ and $b > 0$, or $g_i = e_k + e_{n+1} = f_k$ for some $k \in \{1, \ldots, n\}$. Applying now Corollary 2.13, analogously as in the case of f_i we get that $\ker(g_i)$ cannot be one-complemented in l_Φ. This contradiction completes the proof. □

Theorem 3.3. *Suppose that an Orlicz function ϕ satisfies condition (S), and suppose that there are $2 < p < \infty$ and $0 < u_0 < 1/2$ such that for $u \in [0, u_0]$*

$$\phi(u) = u^p,$$

and $[0, u_0]$ is the largest interval having this property. Then there exists a subspace $Y \subset l_\phi$ of finite codimension which cannot be represented as an intersection of one-complemented hyperplanes.

Proof. Take $0 < f < 1$, $2 < p < \infty$ and $n \in \mathbb{N}$ such that $d = 1/(1 + nf^p)^{1/p} < u_0$. Setting

$$f_i = e_i + fe_{n+1} \quad \text{for } i = 1, \ldots, n,$$

let $Y = \bigcap_{i=1}^n \ker(f_i)$. We shall show that Y is one-complemented in l_ϕ applying Theorem 2.6. Observe that

$$f/(1+nf^p)^{1/p} < 1/(1+nf^p)^{1/p} < u_0.$$

Hence it is easy to verify that

$$z = -f\left(\sum_{j=1}^n e_j\right) + e_{n+1}$$

has norm $(1+nf^p)^{1/p}$. Setting for $i = 1, \ldots, n$, $b_i = f^{p-1}/(1+nf^p)$, it is easy to check that the equality in Theorem 2.10 is satisfied for $t \in [0, 1/\|z\|]$, and thus Y is one-complemented in l_ϕ.

In order to show that Y cannot be represented as an intersection of one-complemented hyperplanes, at first we prove that $\ker(f_i)$ is not one-complemented for any $i = 1, \ldots, n$. But the latter is clear in view of Corollary 2.16 and the inequality $u_0 < 1/2 \leqslant 1/\|e_1 + e_2\|$.

Now, suppose on the contrary that $Y = \bigcap_{i=1}^n \ker(g_i)$, where $\ker(g_i)$ is one-complemented for any $i = 1, \ldots, n$. By Theorem 2.2, applied for $i = 1, \ldots, n$ to $\ker(g_i)$, $g_i = r(g_i)$ and each g_i has at most two coordinates different from zero. Since for $i = 1, \ldots, n$, $g_i = \sum_{j=1}^n a_{ij} f_j$ for some $a_{ij} \in \mathbb{R}$, $g_i = b(-e_k + e_l)$ for some $k, l = 1, \ldots, n$, $b > 0$ or $g_i = a_{ik}e_k + a_{ik} f e_{n+1}$ for some $k = 1, \ldots, n$. We observe that Y contains all elements y of the form $y = Cz$, $C \in \mathbb{R}$. It follows that there exists $k = 1, \ldots, n$ such that at least for some $i = 1, \ldots, n$, $g_i = a_{ik}e_k + a_{ik} f e_{n+1} = a_{ik} f_k$. Thus $\ker(g_i)$ cannot be one-complemented, which contradicts the assumption and finishes the proof. □

4. Characterization of l_p-spaces in the class of l_Φ-spaces

By a result of Calvert and Fitzpatrick [9], if X is a Banach lattice with a Schauder basis (e_i) such that $e_i \wedge e_j = 0$, and if for any $a, b \neq 0$ and $j, k \in \mathbb{N}$, $j \neq k$, $\ker(a(e_j)^* + b(e_k)^*)$ is one-complemented in X, then X is isometric to l_p or c_0. Here $(e_j)^*$ and $(e_k)^*$ denote the corresponding to e_j and e_k biorthogonal functionals. A similar result holds true in l_Φ without the separability assumption, as it is proved below.

Theorem 4.1. *Suppose that for any $f = (f_n) \in l_{\Phi^*}$ with exactly two coordinates different from zero, $\ker(f)$ is one-complemented in l_Φ. Then there exists $p \in (1, \infty)$ such that*

$$\phi_n(t) = t^p \quad for\ t \in [0, 1],\ n \in \mathbb{N},$$

that is the identity operator is an isometry from l_Φ to l_p.

Proof. Set $A = 1/\|e_1 + e_2\|$. By our assumptions and Corollary 2.13, for any $a \in (0, 1]$ there exists $b_a > 0$ such that for all $u \in [0, A]$

$$\phi_1(au) = b_a a \phi_2(u). \tag{4.1}$$

By Lemma 2.9, there exist $K_1 > 0$ and $p \in (1, \infty)$, such that $\phi_1(u) = K_1 u^p$ for $u \in [0, A]$. Notice that $p > 1$, since $\phi_n'(0) = 0$, $n \in \mathbb{N}$, by condition (s). Hence for any $a \in (0, 1)$ and $u \in [0, A]$

$$a^p \phi_1(u) = \phi_1(au) = b_a a \phi_2(u) = (b_a/b_1) a \phi_1(u)$$

and consequently, $b_a = a^{p-1} b_1$. Notice that by Corollary 2.13 for any $u < 1/\|ae_1 + e_2\|$, $\phi_1(au) = Ca\phi_2(u)$. But, by (4.1), $C = b_a$. Hence for any $u < 1/\|ae_1 + e_2\|$ and $au < A$, we have

$$K_1(au)^p = \phi_1(au) = b_1 a^p \phi_2(u)$$

which gives

$$\phi_2(u) = (K_1/b_1) u^p.$$

Since for any $u \in [0, 1)$ one can choose $a \in (0, 1)$ satisfying both $u < 1/\|ae_1 + e_2\|$ and $u < A/a$, the above equality holds true for all $u \in [0, 1]$. Finally in view of $\phi_2(1) = 1$, $\phi_2(u) = u^p$ for $u \in [0, 1]$. In a similar way we get that $\phi_n(u) = \phi_2(u) = u^p$ for any $n \in \mathbb{N}$ and $u \in [0, 1]$, as required. $\square$

From the proof of Theorem 4.1 it is easy to deduce the following result.

Corollary 4.2. *Suppose that for some $j, k \in \mathbb{N}$, $k \neq j$ and any $a, b \neq 0$, $\ker(ae_j + be_k)$ is one-complemented in l_Φ. Then there exists $1 < p < \infty$ such that for $u \in [0, 1]$, $\phi_j(u) = \phi_k(u) = u^p$.*

Lemma 4.3. *Let Φ be a Musielak–Orlicz function. Suppose that there is $u_0 > 0$ such that for all $i \in \mathbb{N}$ and $u \in [0, u_0]$, $\phi_i(u) = C_i \phi_1(u)$ for some $C_i > 0$ independent of u. Let for $n \geqslant 3$,*

$$y_n = -e_1 + \sum_{j=2}^{n} \frac{1}{n-1} e_j.$$

Then $\lim_n \|y_n\| = 1$.

Proof. Fix $a > 1$ and $\varepsilon < 1 - \phi_1(1/a)$. Choose $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$, $1/((n-1)a) < u_0$ and $\phi_1'(1/((n-1)a))/\phi_1(u_0) < \varepsilon$. Since $\phi_i(1) = 1$, $\phi_i'(1/((n-1)a)) < \varepsilon$ for all $i \in \mathbb{N}$. Observe that for $n \geqslant n_0$, by convexity of ϕ_n and the Lagrange Theorem,

$$\begin{aligned}
\rho_\Phi(y_n/a) &= \phi_1(1/a) + \sum_{j=2}^{n} \phi_j(1/((n-1)a)) \\
&\leqslant \phi_1(1/a) + (n-1) \max_{j=2,\ldots,n} \phi_j(1/((n-1)a)) \\
&\leqslant \phi_1(1/a) + (n-1)\phi_{j_0}(1/((n-1)a)) \\
&\leqslant \phi_1(1/a) + (n-1)\phi_{j_0}'(1/((n-1)a))/((n-1)a) \\
&\leqslant \phi_1(1/a) + (1/a)\varepsilon < 1,
\end{aligned}$$

where $j_0 \in \{2, \ldots, n\}$. Hence for $n \geqslant n_0$, $\|y_n\| \leqslant a$. Since $\|y_n\| > 1$ for any $n \geqslant 2$, we get that $\|y_n\|$ tends to one and the proof is complete. $\square$

Theorem 4.4. *Let* $f_n = \sum_{j=1}^{n} e_j$, *and suppose that* $\ker(f_n)$ *is one-complemented in* l_Φ *for sufficiently large* n. *Then*

$$\phi_i(t) = t^2 \quad for\ t \in [0,1],\ i \in \mathbb{N},$$

that is the identity operator from l_Φ *to* l_2 *is an isometry.*

Proof. Assume that there exists n_0 such that $\ker(f_n)$ is one-complemented for every $n \geqslant n_0 \geqslant 3$ and let $Q_n \in \mathcal{P}(l_\Phi, \ker(f_n))$ be a projection of norm one. Let $z_n = (z_{ni})$ be given for Q_n and $\ker(f_n)$ by Lemma 1.2. Fix $u \in [0, 1/2]$. Since $\|e_1 + e_j\| \leqslant 2$, we find $t_j \geqslant 0$ and $k > n$ such that $w_j = -ue_1 + ue_j + t_j e_k,\ j = 2, \ldots, n$, has norm one. It is clear that $w_j \in \ker(f_n)$. Consequently, by Corollary 1.6, for any $j \leqslant n$ and $0 \leqslant u \leqslant 1/2$

$$-z_{n1}\phi_1'(u) + z_{nj}\phi_j'(u) = 0,$$

which gives

$$\phi_j'(u) = (z_{n1}/z_{nj})\phi_1'(u). \tag{4.2}$$

Now we shall show that for any $a, b \geqslant 0$, $a + b \leqslant 1/2$,

$$\phi_1'(a+b) = \phi_1'(a) + \phi_1'(b). \tag{4.3}$$

Taking any $a, b \geqslant 0$, $a + b \leqslant 1/2$ and setting

$$w = 2(ae_1 + be_2 - (a+b)e_3),$$

$\rho_\Phi(w/2) = \phi_1(a) + \phi_2(b) + \phi_3(a+b) \leqslant 2(a+b) = 1$, which implies that $\|w\| \leqslant 2$. Let $n \geqslant n_0$. Observe that for any $t \in [0,1]$ we can find $b_t > 0$ such that

$$t/\|w\| = 1/\|w_t\|,$$

where

$$w_t = 2ae_1 + 2be_2 - 2(a+b)e_3 + b_t e_{n+1}.$$

Note that $w_t \in \ker(f_n)$. Hence, by Corollary 1.6, for any $t \in [0,1]$

$$z_{n1}\phi_1'(t2a/\|w\|) + z_{n2}\phi_2'(t2b/\|w\|) = z_{n3}\phi_3'(t2(a+b)/\|w\|).$$

Since $\|w\| \leqslant 2$, taking $t = \|w\|/2$, and applying (4.2) we get (4.3). Now we show that (4.3) holds true for any $a, b \geqslant 0$ with $a + b < 1$. Note that by (4.2) the assumptions of Lemma 4.3 are satisfied with $u_0 = 1/2$. Consequently, we can choose $n \in \mathbb{N}$ such that $\ker(f_n)$ is one-complemented and

$$a + b < 1/\|y_n\|$$

where y_n is defined by the formula in Lemma 4.3. By Corollary 1.6, analogously as above, we get for any $t \in [0,1]$

$$z_{n1}\phi_1'(t/\|y_n\|) = \sum_{j=2}^{n} \phi_j'(t/((n-1)\|y_n\|))z_{nj}.$$

Since for $n>1$, $t/(\|y_n\|(n-1)) \leqslant t/(n-1) \leqslant 1/2$, by (4.2)

$$\phi_1'(t/\|y_n\|) = (n-1)\phi_1'(t/((n-1)\|y_n\|)).$$

Taking $t=(a+b)\|y_n\|$, we get

$$\phi_1'(a+b) = (n-1)\phi_1'((a+b)/(n-1)),$$

and then taking $a\|y_n\|$ or $b\|y_n\|$ as t we also obtain

$$\phi_1'(a) = (n-1)\phi_1'(a/(n-1)) \quad \text{and} \quad \phi_1'(b) = (n-1)\phi_1'(b/(n-1)).$$

Combining the above equations with (4.3) we get that for any $a, b \geqslant 0,\ a+b<1$, it holds

$$\begin{aligned}\phi_1'(a+b) &= (n-1)\phi_1'((a+b)/(n-1))\\ &= (n-1)\phi_1'(a/(n-1)) + (n-1)\phi_1'(b/(n-1))\\ &= \phi_1'(a) + \phi_1'(b).\end{aligned}$$

Consequently, for $t \in [0,1]$

$$\phi_1'(t) = t\phi_1'(1).$$

Analogously as above, we can show that the above equality holds true for any function ϕ_j. Finally in view of the assumptions $\phi_j(1)=1$ and $\phi_j(0)=0$, we obtain that $\phi_j(t)=t^2$ for $t \in [0,1]$. □

Theorem 4.5. *Let ϕ be an Orlicz function and let $f_n = \sum_{j=1}^n e_j$. Then* $\ker(f_n)$ *is one-complemented in l_ϕ for $n \geqslant 3$, if and only if there exists $C>0$ such that*

$$\phi(t) = Ct^2, \quad t \in [0, 1/\|y_n\|], \tag{4.4}$$

where $y_n = -e_1 + \sum_{j=2}^n (1/(n-1))e_j$.

Proof. Suppose that for some $n \geqslant 3$, $\ker(f_n)$ is one-complemented in l_ϕ. In a similar way as in the proof of Theorem 4.4, we obtain that for $a, b \geqslant 0$, $a+b \leqslant 1/\|y_n\|$,

$$\phi'(a+b) = \phi'(a) + \phi'(b).$$

Hence we conclude that for any $u \in [0, 1/\|y_n\|]$,

$$\phi'(u) = u\|y_n\|\phi'(1/\|y_n\|),$$

which in view of $\phi(0)=0$ immediately implies that for $u \in [0, 1/\|y_n\|]$,

$$\phi(u) = Cu^2.$$

Now suppose that (4.4) holds true. We claim that for any $0 \neq z \in \ker(f_n)$

$$\max_{j=1,\ldots,n} \{|z_j|/\|z\|\} \leqslant 1/\|y_n\|. \tag{4.5}$$

Indeed, let $c = \|y_n\|$ and

$$A = \{u = (u_1, \ldots, u_{n-1}) \in [0,1]^{n-1} : \sum_{j=1}^{n-1} u_j = 1\}$$

and let $f : [0,1]^{n-1} \to \mathbb{R}$ be defined by

$$f(u) = \sum_{j=1}^{n-1} \phi(u_j/c).$$

We show first that f attains a conditional minimum on A at $w_n = \frac{1}{n-1}(\sum_{j=1}^{n-1} e_j)$. We will apply the Lagrange multiplier method. Let us define for $\lambda \in \mathbb{R}$,

$$g_\lambda(u) = f(u) + \lambda\left(\sum_{j=1}^{n-1} u_j - 1\right).$$

Consider for $j = 1, \ldots, n-1$ the system of equations

$$\frac{\partial g_\lambda(u)}{\partial u_j} = (1/c)\phi'(u_j/c) + \lambda = 0 \quad \text{and} \quad \frac{\partial g_\lambda(u)}{\partial \lambda} = \sum_{j=1}^{n-1} u_j - 1 = 0.$$

It is easy to see that $w_n = \frac{1}{n-1}(\sum_{j=1}^{n-1} e_j)$ is the only solution of this system. We need still to check the value of f at the boundary of A. We will apply the induction argument. By our assumptions for $n = 3$,

$$f(w_3) = 2\phi(1/(2c)) < \phi(1/c) = f(1,0) = f(0,1),$$

which shows that the conditional minimum is attained at w_3. Now let $n > 3$ and take any w from the boundary of A. Let k denote the number of nonzero coordinates of w. Clearly, $k < n-1$. By the induction hypothesis and the convexity of ϕ,

$$\begin{aligned} f(w) \; > \; f(w_{k+1}) &= k\phi(1/(kc)) = (n-1)\left(\frac{k}{n-1}\right)\phi\left(\frac{(n-1)/k}{(n-1)c}\right) \\ &> (n-1)\phi(1/((n-1)c)) = f(w_n). \end{aligned}$$

Hence f cannot attain a conditional minimum on the boundary of A, so it has to attain it at w_n. Now, let $0 \neq z \in \ker(f_n)$. Since the space l_ϕ is symmetric, without loss of generality, we assume that for $j = 1, \ldots, n-1$, $|z_j| \leqslant 1 = -z_n$. Thus

$$1 = \sum_{j=1}^{n-1} z_j \leqslant \sum_{j=1}^{n-1} |z_j|.$$

By the previous part of the proof,

$$1=\sum_{j=1}^{n-1}\phi(1/((n-1)\|y_n\|))+\phi(1/\|y_n\|)$$
$$=f(w_n)+\phi(1/\|y_n\|)\leqslant\sum_{j=1}^{n-1}\phi(|z_j|/\|y_n\|)+\phi(1/\|y_n\|).$$

Hence $1/\|z\|\leqslant 1/\|y_n\|$, which shows (4.5).

Finally, we show that $P\in\mathcal{P}(l_\phi,\ker(f_n))$ given by

$$Px=x-f_n(x)w_{n+1},\quad x\in l_\phi,$$

has norm one. Now $\sum_{j=1}^{n} z_j/\|z\|=0$, and thus by (4.5) and (4.4)

$$N(z)(w_{n+1})=\left(\sum_{j=1}^{n}\operatorname{sgn}(z_j)\phi'(|z_j|/\|z\|)\right)\Big/ n=(2C/n)\sum_{j=1}^{n}z_j/\|z\|=0.$$

By Corollary 1.6, $\|P\|=1$ and the proof is complete. □

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